CONJUGATED TRANSIENT HEAT TRANSFER IN PIPE FLOW

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THESIS

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by

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TABLE OF SYMBOLS

Symbol		Quantity	Units
A_m	=	cross sectional area of metal	ft^2
A_f	=	cross section area of fluid	ft^2
a	=	inner radius of pipe wall	ft
a	=	half width of channel (slab geometry)	ft
a	=	$b_1 b_2 \eta$, used in equations (IV-29) and (IV-31)	hr^{-1}
B_f	=	$^{ha}/K_f$ = Fluid Biot number defined by equation (V-33)	
B_f^*	=	$^{2ha}/K_F$ = Fluid Biot number defined by equation (V-62)	
B_m	=	$^{hb}/K_m$ = pipe wall Biot number defined by equation (V-33)	
b	=	outer radius of pipe	ft
b	=	thickness of wall (slab geometry)	ft
b_1	=	$^{hP}/\rho_f C_f A_f$ = parameter defined by equation (IV-2)	hr^{-1}
b_2	=	$^{hP}/\rho_m C_m A_m$ = parameter defined by equation (IV-5)	hr^{-1}
c_f	=	Fluid specific heat	$^{Btu}/_{lbm}{}^{o}F$
c_m	=	pipe wall specific heat	$^{Btu}/_{lbm}{}^{o}F$
C_i	=	coefficient defined by equation (IV-57), i = 0, 1	
C_i^*	=	coefficient defined by equation (IV-67), i = 0, 1	

h	=	heat transfer coefficient at fluid-wall interface	Btu/ft^{2}hr^{o}F
$\tilde{h}$	=	heat transfer coefficient defined by equation (II-15)	Btu/ft^{2}hr^{o}F
h^*	=	heat transfer coefficient defined by equation (II-9)	Btu/ft^{2}hr^{o}F
$h(\ell, n;t)$	=	Function defined by equation (IV-45)	
$I_n(x)$	=	modified Bessel function of first kind of order n	
I_n	=	complex integral defined in Appendix A equation (5)	
$J_n(x)$	=	Bessel function of first kind of order n	
K_{ma}	=	pipe wall axial conductivity	Btu/ft hr^{o}F
K_{mt}	=	pipe wall transverse conductivity	Btu/ft. hr^{o}F
K_m	=	pipe wall isotropic conductivity	Btu/ft. hr^{o}F
K_{fa}	=	fluid axial conductivity	Btu/ft. hr^{o}F
K_{ft}	=	fluid transverse conductivity	Btu/hr ft ^{o}F
K_f	=	fluid isotropic conductivity	Btu/hr ft ^{o}F
k_i	=	coefficients defined by equation (IV-40), i = 0, 1, 2, 3, 4	
k_i^*	=	coefficients defined by equation (IV-42), i = 1, 2, 3, 4, 5	
$K_n(x)$	=	modified Bessel function of second kind of order n.	
P	=	perimeter	ft
q	=	$(s/\alpha)^{1/2}$	ft^{-1}
r	=	radial coordinate	ft
s	=	Laplace transform parameter	hr^{-1}

T	=	pipe wall temperature	^{o}F
T^*	=	radially averaged pipe wall temperature	^{o}F
t	=	Time	hr
u_i	=	temperature functions used in Section III	^{o}F
V	=	fluid velocity	ft/hr
$W(x, y, t)$	=	temperature function defined by equation (III-21)	^{o}F
$W(r, x, t)$	=	temperature function defined by equation (III-92)	^{o}F
x	=	axial coordinate	ft
y	=	transverse coordinate	ft
$Y_n(x)$	=	Bessel function of second kind of order n	

GREEK LETTERS

α_f	=	fluid thermal diffusivity	ft^2/hr
α_m	=	pipe wall thermal diffusivity	ft^2/hr
ϵ	=	$^1/V$ (used in Section IV)	hr/ft
η	=	x/V (used in Section IV)	hr
θ	=	fluid temperature	^{o}F
θ^*	=	radially averaged fluid temperature	^{o}F
ξ	=	$(\eta - t)/\epsilon$ (used in Section IV)	ft
ρ_f	=	fluid density	lbm/ft^3
τ	=	$\eta - t$ (used in Section IV)	hr
ϕ	=	fluid temperature function at interface	^{o}F

χ = function defined by equation (V-43)

ψ = pipe wall temperature function at interface ^{o}F

Also, additional notations are occasionally introduced for temporary use in the analytical developments which follow. In such cases, the new notations will be adequately defined in the text. This also implies that the same symbol may have different significance in different portions of the text. There should be no confusion arising from this practice. In particular, different notation is frequently introduced for dummy variables of integration. Also, the symbols β_n, γ_n, λ_n, μ_n, etc., used as eigen values, do not appear in the foregoing list.

We also remark that equations are numbered serially in each section. In referring to equations, the serial number alone will indicate the correspondingly numbered equation in the <u>same</u> section as the reference; otherwise the section number will also appear. Thus, in Section V, for example, the reference (12) is to equation (V-12) and the reference (II-7) is to equation (II-7), that is, to equation 7 of Section II. A reference such as IID2 is to the text of subsection IID2 and not to an equation.

ABSTRACT

Transient temperature distribution in axisymmetric systems of incompressible fluid flow through semi-infinite, externally insulated cylindrical pipes is considered under the assumption of constant uniform velocity and constant material properties. The general problem, including all significant heat transfer mechanisms, except radiation, is reduced to an integral equation containing only one unknown function, namely an interface temperature as a function of time and axial position. By neglecting one or more heat transfer mechanisms, several other formulations are obtained. These are analyzed so as to provide approximate and closed form solutions not previously available. Similar formulations and results are obtained for the case of fluid between parallel plates.

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I. INTRODUCTION

A. PRELIMINARY REMARKS

The problem of transient temperature fields in systems of fluid flow through circular pipes or through channels formed by parallel plates is of considerable practical and theoretical importance. On the practical side, one may consider, for example, the problem of the calculation of thermal stresses in main coolant piping systems of nuclear power plants in which abrupt changes in power output may cause very rapid variation in the temperature of the coolant fluid leaving the reactor. In such cases one may be concerned with the time history of the coolant temperature at some critical point, such as a pump or valve, downstream from the reactor.

From a theoretical point of view, the problems implied by such pipe flows are of great interest. They involve two dependent variables, the temperatures of the fluid and of the pipe wall, each of which is a function of three independent variables: time and two space coordinates. These functions must satisfy appropriate partial differential equations and external boundary and initial conditions and must also conform to each other in an appropriate way at their common interface. Such problems are called "conjugated." Few solutions for such problems are given in the literature. Those which were previously known usually involve drastic simplifications such as considering only steady state

behavior or restricting space variation to one dimension, or considering the pipe to have infinite thickness, or neglecting important mechanisms of heat transfer, or a combination of these.

Accordingly it is important to consider such problems in as generalized a form as possible both for the purpose of obtaining compact alternate formulations and also for the purpose of obtaining closed form and approximate solutions to problems in which the degree of simplification is less than that involved in previously known solutions. It is believed that the work reported herein is successful in both regards.

B. STATEMENT OF THE GENERAL PROBLEM

It is appropriate at this point to make an explicit statement of the general problem which will be considered herein. We are concerned with the flow of an incompressible fluid axially down the length of a semi-infinite cylindrical pipe whose cross section is a circular annulus of constant finite wall thickness. It is presumed that axisymmetrical conditions prevail so that the space variables are radius r and axial position x from the accessible end of the pipe. The outside of the pipe is presumed to be perfectly insulated so that there is no transfer of heat through this surface. (This condition is approximately satisfied in many cases of technical importance.)

It is assumed that the physical properties of both the fluid and the pipe wall are constant -- that is, they do not vary with respect to time,

temperature, or position. We presume that the thermal conductivity of the materials involved is orthotropic; that is, we provide for different values of conductivity in the radial and axial directions. This permits later specialization in different applications, either to the case of isotropic conductivity or to the cases where one or more conducting mechanisms are suppressed. Subject to this introduction of orthotropic conductivity, it is assumed that Fourier's law of heat conduction holds in each medium. In most of the work done herein, we provide for an interface surface heat transfer coefficient to account for surface resistance or other effects. This coefficient may be taken as infinite if there is no such resistance. However, in all cases it is assumed to be a constant which does not depend on time, position, or temperature.

We have assumed the nature of the fluid flow, namely, we have assumed that the velocity of each element of fluid is the same, a constant velocity V in the axial direction x. This is frequently called slug flow. It is recognized that this is a particular case which is at variance with the flow conditions which would actually obtain under any circumstances. However, it is not far removed from the flow conditions which would prevail with fully turbulent flow and it does provide a necessary simplification which has permitted us to obtain useful results.

We have confined attention to one case of temperature variation which can be described as follows. It is presumed that the fluid has been entering the pipe at zero (or reference) temperature for a

solution for the case of cylindrical geometry. In fact, in some cases, when certain simplifying assumptions and procedures are introduced, the two cases may become one and the same.

C. SURVEY OF LITERATURE

The study of heat transfer problems in systems of fluid flow through pipes dates back to 1883 when one of the most celebrated problems of heat transfer was solved by L. Graetz [11].* In Graetz's problem the velocity field is taken to be known, and heat transfer takes place under steady state conditions with pipe wall temperature distribution specified a priori. In recent years other problems of fluid flow have been treated by many authors. These studies include such effects as heat transfer by axial conduction and mechanical heat generation due to friction, i. e., viscous dissipation. There is an abundance of literature in this area of heat transfer engineering. However, these problems are concerned solely with behavior of the fluid; the action of the containment is considered only in the specification of boundary conditions for the fluid.

However, the transient case of temperature field in systems of flow through pipes or ducts when the wall thermal state is not specified a priori, and it is required that its effect on the temperature field in the fluid be accounted for, clearly falls in the class of conjugated

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*Numbers in square brackets refer to the Bibliography, page 154

problems of heat transfer. These conjugated problems are more involved inasmuch as the mutual thermal effects of the wall and the fluid must be accounted for. Accordingly their treatment presents serious mathematical difficulties. Even when the velocity field is assumed to be known, the thermal problems are still of great difficulty, mainly because of the coupling that is introduced by the condition at the common boundary of the fluid and the wall.

At present there are numerous publications dealing with conjugated problems of heat transfer. In all cases, however, they seem to be particularized and as yet no general method has been obtained for treatment of these problems.

An early treatment of conjugate problems of heat transfer was given by T. L. Perelman [25] who considered steady state heat transfer to a fluid flowing around a body containing internal heat sources. He also considered the asymptotic solution to the type of integral equation that occurs in the analysis of such conjugated problems: namely, a Fredholm equation of second kind with singular kernel and infinite range. He considered slip flow around a body and also considered the laminar boundary layer flow over a thin plate. Perelman neglected the axial heat conduction in the fluid.

N. A. Avdonin [4] considered transient behavior in a similar case but with no heat sources present. He also neglected the axial conduction in the fluid. He solved the problem by using Green's functions and Laplace transforms. He treated the conjugated problem by

reducing it to a Volterra integral equation which he solved using Laplace transforms.

D. E. James and N. G. William [17] considered the steady state problems of Poiseuille - Couette flow between parallel planes. In this work the boundary planes were taken to be of finite length in the direction of flow and of finite thickness perpendicular to the direction of flow. These authors assumed an arbitrary temperature distribution at the fluid-wall interface and solved the energy equations separately for each material for a constant interface temperature distribution. Then they used Duhamel's superposition integral to obtain the general solutions in terms of the arbitrary interface temperature distribution. Next applying the continuity condition for the heat flux they obtained an integro-differential equation for the interface temperature distribution. They solved this by the method of collocation.

Recently A. V. Luikov, V. A. Aleksashenko, and A. A. Aleksashenko [22], considered conjugated steady state heat transfer in circular and plane tubes for the case of developed Poiseuille velocity distribution allowing for mechanical energy dissipation. They state that general methods have been developed for both steady and unsteady heat transfer, but they give no example or discussion for other than steady state cases. They essentially reduce the problem to a singular integral equation for the unknown temperature at the fluid-wall interface. This is a very interesting paper.

E. M. Sparrow and F. N. DeFarias [33] investigated the laminar flow through parallel plate channels under the assumption of sinusoidally varying fluid inlet temperature. This work is in the spirit of the new results to be reported herein inasmuch as a solution for sinusoidally varying end temperature can be related to that for step change in end temperature and conversely. Accordingly it is important to delineate clearly what simplifying assumptions they employed. They neglected axial conduction in both fluid and pipe and they assume pipe temperature varies with axial position but not with transverse position. They considered constant fluid velocity as we do herein. It happens that these simplifying assumptions do not coincide with any set treated in this thesis.

V. S. Arpaci [3] considered a transient problem of the type in which we are interested. Assuming as we do, the case of slug flow, he uses the term "rod" to describe what we call the fluid. He shifts viewpoint in an obvious and nonessential way by considering the pipe to be moving at constant velocity with its contents at zero velocity. Axial conductivity is neglected in both substances. Radial conductivity is accounted for in the fluid. Radial conductivity of the pipe wall is taken as infinite. The transient effect is due to sudden and uniform heat generation in the contents. Like ours, his pipe has perfect external insulation. He used the method of finite Hankel transforms to reduce the problem to the solution of an integro-differential equation

which was solved by Laplace transforms. For large values of time an asymptotic behavior of the temperature function was found, again using Laplace transforms.

Other significant contributions to the study of conjugated heat transfer problems have been made by M. D. Kelleher and K. T. Yang [19], R. Siegel [34], and R. Siegel and J. M. Savino [35].

F. E. Moreland [23] employed numerical inversion of Laplace transformation to deal with heat exchanger problems which are similar to our problems, except for suppression of all heat diffusion in the fluid. D. D. Hiep [13] extended this work significantly by including the effect of axial heat diffusion. (His treatment is for porous media, but our formulation could be similarly generalized by use of some fictitious material and geometrical properties.) His study contains a large bibliography. The problem considered in Section VI hereof is a special case of that considered by Hiep. We are able to obtain analytical results whereas Hiep's analysis is purely numerical.

Two earlier solutions which provide significant points of departure should also be mentioned, even though idealizing assumptions made the analyses very simple. One of these is by W. Munk [24] who assumes identical constant temperature in both pipe and contents at any cross section. He neglects all diffusion of heat and arrives at a solution which repeats the time-temperature history at the entrance point, but with a propagation velocity less than the velocity of the fluid. The second, to which many investigators have made contributions,

is discussed in section 15.3 of [7] where many references are cited. Arpaci [2], page 353 et seq., gives a very compact treatment of this problem in which there is no heat diffusion in either the axial or radial direction.

D. SUMMARY AND OUTLINE OF SUCCEEDING SECTIONS

The following telegraphically brief statements are intended to provide a summary of the content of the succeeding sections.

Section II - Derives basic equations which will be employed in subsequent sections.

Section III - Considers radial and axial heat diffusion in each material. Arrives at integral equation (III-70) for slab geometry and integral equation (III-181) for cylindrical geometry. No solution of these equations is attempted.

Section IV - Radially averaged temperatures are considered in each material. An ordinary perturbation solution is constructed which is valid "behind the (temperature) front." A singular perturbation solution is developed which gives variation ahead of the front. A scheme for matching these at the front is given.

Section V - By averaging the fluid temperature in the transverse direction and neglecting axial conduction in the pipe wall, closed form solutions are obtained in the case of slab geometry: equations (V-39, 40). The case of cylindrical geometry turns out to differ only as regards the definition of a certain auxiliary function so that the solution has the same expression as for the case of slab geometry.

Section VI - By taking radial averages for both fluid and solid temperatures and by neglecting axial conduction in the solid, a simplified version of the preceding problem is obtained. (This may also be obtained by neglecting axial conduction in the solid in the case given in section IV). A closed form solution is given: equation (VI-21) and (VI-23).

Section VII - Here the problem is greatly simplified by assuming (in effect) that there is no radial variation in temperature. Axial conduction in both materials is considered, however; (i. e., we take average fluid temperature equal to average solid temperature at any section and employ the equations of section IV). The resulting solution, (VII-10) is not new but is included here for completeness and comparison purposes.

Section VIII - This section includes discussion and recommendation for further study.

II. FUNDAMENTAL EQUATIONS

A. GENERAL REMARKS

All the analysis presented in this thesis relates to a definite physical problem which will be described shortly. The real physical problem is so complicated as to preclude the possibility of dealing with it without making certain idealizations. In what follows immediately, the problem will be described in mathematical terms and the idealizations which are common to all the subsequent analysis herein will be delineated. The simplified problem which results is still of great complexity. It will be further discussed in section III. No solution of this problem will be presented but it will be shown that the problem can be cast in a more compact manner which gives some promise of being amenable to analysis.

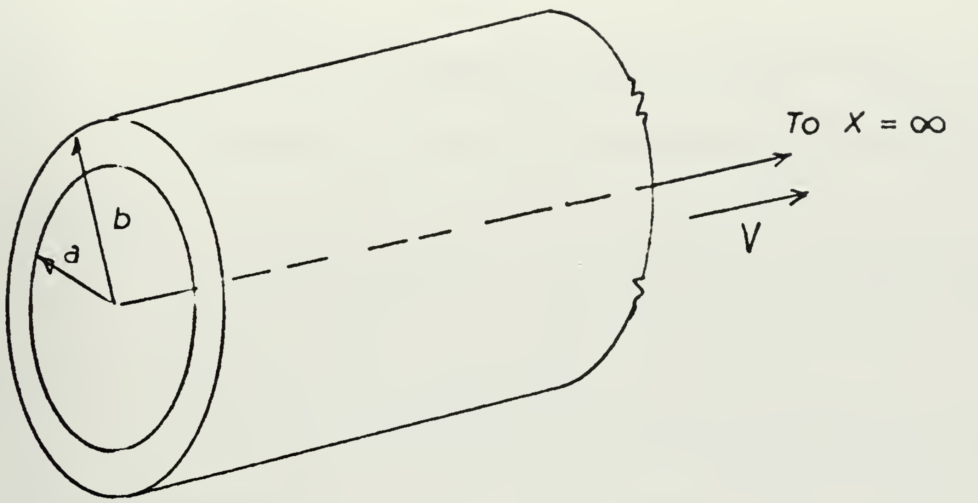
In order to obtain additional results, it has been found necessary to make additional idealizations or simplifications. These lead to several simplified problems which are specifically delineated and dealt with in sections IV through VII. For all of this work, however, the derivation given immediately below and the equations which result therefrom are fundamental.

1. Physical Model

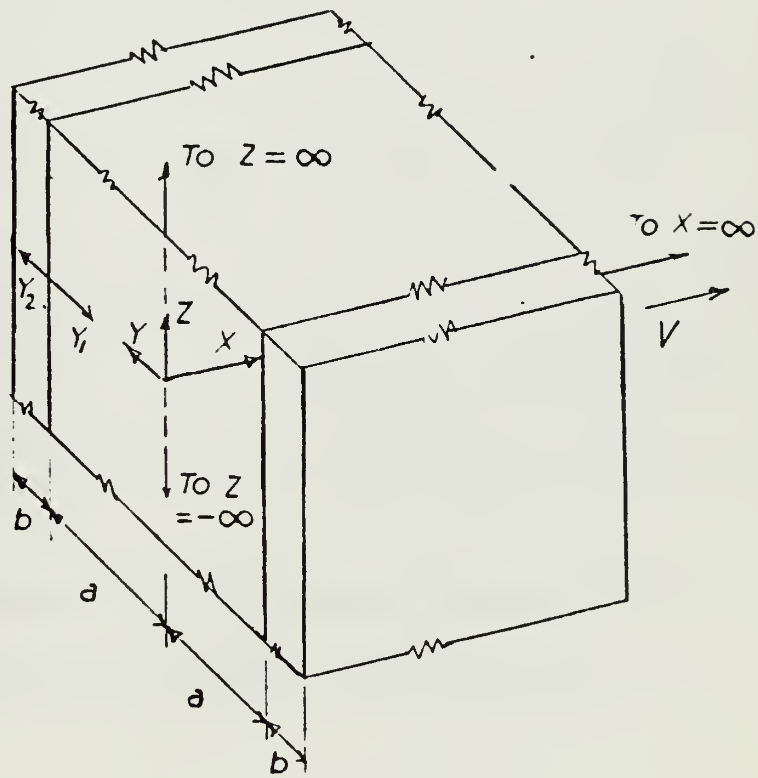
The basic physical problem of interest is that of heat transfer taking place between a conduit and the fluid flowing therein. The former

is perfectly insulated on its exterior surface. It may be either a circular pipe with uniform wall thickness or a duct composed of two parallel plates. (In what follows these two cases will be referred to as "cylindrical geometry" and "slab geometry," respectively. It is assumed that the fluid entering the conduit has had constant temperature for sufficient time that the conduit and its fluid contents have reached this constant temperature throughout. Then the temperature of the fluid entering the conduit is suddenly changed by a finite amount and is thereafter maintained at the new temperature. (The end surfaces of the conduit also experience the same step change in temperature.) It is required to predict the unsteady temperature fields in both the fluid and the conduit for all locations at any subsequent time.

Figure 1 illustrates the two geometries which are considered. That for the cylindrical geometry is self-explanatory, except to remark that there is no variation whatsoever with respect to angular coordinate. The case of slab geometry involves identical parallel plates which are semi-infinite in extent in the x direction, which is the direction of fluid flow, and are infinite in the z direction. It is assumed that there is no variation whatsoever with respect to the coordinate z of any quantity. In making a mathematical formulation of the problem, it will be convenient to take advantage of the symmetry with respect to the plane $y = 0$; for this reason the auxiliary coordinates y_1 and y_2 are used as indicated.



a - cylindrical geometry



b - slab geometry

Figure 1: Geometry and Dimensions of Conduits

2. <u>Assumptions</u>

a. The fluid is considered to be incompressible.

b. The fluid is assumed to be flowing with constant velocity V (slug flow).

c. Material properties are taken to be constant and independent of time, position, and temperature.

d. It is assumed that there may be an additional thermal resistance (possibly due to a very thin layer of foreign material at the interface) the heat transfer coefficient of which is a constant.

e. Viscous dissipation is assumed negligible.

f. Heat transfer by radiation is considered to be negligible.

g. Fourier's law of heat conduction is taken to be valid throughout.

h. The diffusivity appearing in Fourier's law for fluid is assumed to include not only the molecular diffusivity but also an eddy diffusivity which is effective. The combined diffusivity is assumed to satisfy assumption c.

i. There is variation only with respect to time, the axial dimension, and a radial dimension, Figure 1a, or a transverse direction, Figure 1b.

B. DIFFERENTIAL EQUATION

1. Cylindrical Geometry

There is no difficulty in establishing, by customary methods of derivation, the following partial differential equations: (1) which describes the fluid and (2) which describes the solid pipe wall. We have anticipated simplifications which will be made in later sections hereof by considering artificial media in which thermal conductivities in the axial and transverse directions may differ. These conductivities are distinguished by a second subscript, a to denote axial or t to denote transverse. If we are considering isotropic materials (as we do in section III hereof, for example) we take the axial conductivity to be equal to the transverse conductivity. However, if in order to make mathematical headway in a certain analysis we neglect axial conductivity in the solid (say), we can obtain the corresponding mathematical description simply by setting $K_{ma} = 0$.

There should be no difficulty in recognizing the significance of the various terms which appear in equations (1) and (2) below.

$$K_{fa}\frac{\partial^2\theta}{\partial x^2} - \rho_f c_f V\frac{\partial\theta}{\partial x} + K_{ft}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right) = \rho_f c_f \frac{\partial\theta}{\partial t} \tag{1}$$

$$K_{ma}\frac{\partial^2 T}{\partial x^2} + K_{mt}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \rho_m c_m \frac{\partial T}{\partial t} \tag{2}$$

One may observe that (2) can be obtained from (1) by taking $V = 0$, replacing θ by T, and replacing subscript f by subscript m.

2. Slab Geometry

The partial differential equations governing the temperature distribution in the flowing fluid and in the solid duct wall are

$$K_{fa}\frac{\partial^2\theta}{\partial x^2} + K_{ft}\frac{\partial^2\theta}{\partial y_1^2} - \rho_f c_f V\frac{\partial\theta}{\partial x} = \rho_f c_f\frac{\partial\theta}{\partial t} \tag{3}$$

for the fluid, and

$$K_{ma}\frac{\partial^2 T}{\partial x^2} + K_{mt}\frac{\partial^2 T}{\partial y_2^2} = \rho_m c_m\frac{\partial T}{\partial t} \tag{4}$$

for the pipe wall. Observe the use of subscripts 1 and 2 for the coordinate y, consistent with figure 1b.

C. INITIAL AND BOUNDARY CONDITIONS

1. Cylindrical Geometry

a. For Fluid

(1) Initial Condition, $t = 0$. The initial temperature of the moving fluid is considered to be uniform, and without any loss of generality, it may be considered as zero.

$$\theta = 0 \qquad 0 \leqq x < \infty \quad , \quad 0 \leqq r \leqq a \quad , \quad t = 0$$

(2) Boundary Condition at $x = 0$. At this boundary the entering fluid temperature is given a sudden step change to a higher

value thereafter. Without loss of generality, the amount of increase may be taken to be unity.

$$\theta = 1 \qquad x = 0 \quad , \quad 0 \leq r \leq a \quad , \quad t \geq 0$$

(3) Boundary Conditions as $x \longrightarrow \infty$. As x approaches infinity the fluid temperature decreases to its initial value.

$$\theta \longrightarrow 0 \qquad x \longrightarrow \infty \, , \quad 0 \leq r \leq a \quad , \quad t \geq 0$$

(usually, this condition is employed in the form: θ remains bounded as $x \longrightarrow \infty$.)

(4) Boundary Condition at r = 0. It is required that as $r \longrightarrow 0$ the fluid temperature remain finite. This condition may also be stated by requiring that the temperature distribution in the fluid to be symmetric about the center line.

Either

$$\lim_{r \to 0} \theta(r, x, t) < \infty \qquad 0 \leq x < \infty \quad , \quad t \geq 0$$

or

$$\frac{\partial \theta}{\partial r} = 0 \qquad r = 0 \, , \quad 0 \leq x < \infty \quad , \quad t \geq 0$$

b. For the Pipe Wall

(1) Initial Condition, t = 0. The initial temperature of the pipe wall is considered to be uniform and equal to the fluid temperature at the initial time.

$$T = 0 \qquad 0 \leq x < \infty \quad , \quad a \leq r \leq b \quad , \quad t = 0$$

(2) Boundary Condition at x = 0. The pipe wall is assumed to experience the same unit step change of temperature as that of the fluid.

$$T = 1 \qquad x = o \quad , \quad a \leqq r \leqq b \quad , \quad t \geqq o$$

(3) Boundary Condition as $x \rightarrow \infty$. Similarly it is required that the pipe wall temperature approach its initial temperature as x approaches infinity.

$$T \rightarrow o \qquad x \rightarrow \infty \quad , \quad a \leqq r \leqq b \quad , \quad t \geqq o$$

(4) Boundary Condition at r = b. Since it is assumed that the pipe wall is perfectly insulated at the outside surface, then the Fourier heat conduction law requires that

$$\frac{\partial T}{\partial r} = o \qquad o \leqq x < \infty \quad , \quad r = b \quad , \quad t \geqq o$$

c. For the Interface Between Fluid and Wall (r = a)

Continuity of heat flux is required at the interface between fluid and pipe wall.

Contemplating a constant heat transfer coefficient h at this boundary, we may thus write

$$\left\{ \begin{aligned} -K_{ft} \cdot \frac{\partial \theta}{\partial r} &= h\,(\theta - T) \\ -K_{mt} \frac{\partial T}{\partial r} &= h\,(\theta - T) \end{aligned} \right\} \quad o \leqq x < \infty \quad , \quad r = a \quad , \quad t \geqq o$$

It is well to note at this point that if the heat transfer coefficient h approaches infinity, then the above conditions reduce to equality of temperature and heat flux, in the following manner.

$$\left\{ \begin{array}{c} \theta = T \\ K_{ft} \dfrac{\partial \theta}{\partial r} = K_{mt} \dfrac{\partial T}{\partial r} \end{array} \right\} \quad 0 \leqq x < \infty \;, \quad r = a \;, \quad t \geqq 0$$

2. Slab Geometry

The details and physical significance of the choice of initial and boundary conditions for this case are similar to those discussed for the case of cylindrical geometry. We will therefore state them mathematically as follows.

a. For the Fluid

(1) Initial Condition t = 0.

$$\theta = 0 \qquad 0 \leqq x < \infty \;, \quad 0 \leqq y_1 \leqq a \;, \quad t \geqq 0$$

(2) Boundary Condition at x = 0.

$$\theta = 1 \qquad x = 0 \;, \quad 0 \leqq y_1 \leqq a \;, \quad t \geqq 0$$

(3) Boundary Condition as $x \longrightarrow \infty$.

$$\theta = 0 \qquad x \longrightarrow \infty \;, \quad 0 \leqq y_1 \leqq a \;, \quad t \geqq 0$$

(4) Boundary Condition at $y_1 = a$.

$$\frac{\partial \theta}{\partial y_1} = 0 \qquad 0 \leqq x < \infty \;, \quad y_1 = a \;, \quad t \geqq 0$$

(Cf. figure 1b; the symmetry about the face $y_1 = a$ provides this condition.)

b. For the Wall

(1) Initial Condition, $t = 0$.

$$T = 0 \qquad o \leqq x < \infty \quad , \quad o \leqq y_2 \leqq b, \quad t = o$$

(2) Boundary Condition at $x = 0$.

$$T = 1 \qquad x = 0 \quad , \quad 0 \leqq y_2 \leqq b \quad , \quad t \geqq o$$

(3) Boundary Condition as $x \longrightarrow \infty$.

$$T = 0 \qquad x \longrightarrow \infty \quad , \quad 0 \leqq y_2 \leqq b \quad , \quad t \geqq o$$

(4) Boundary Condition at $y_2 = b$.

$$\frac{\partial T}{\partial y_2} = 0 \qquad o \leqq x < \infty \quad , \quad y_2 = b \quad , \quad t \geqq o$$

c. For the Interface Between Fluid and Wall $y_1 = y_2 = 0$

$$\left.\begin{cases} -K_{mt} \dfrac{\partial T}{\partial y_2} = h\,(\theta - T) \\[2ex] K_{ft} \dfrac{\partial \theta}{\partial y_1} = h\,(\theta - T) \end{cases}\right\} o \leqq x < \infty \quad , \quad y_1 = y_2 = o \, , \quad t \geqq o$$

Observe that the algebraic signs given here are a consequence of the fact that y_1 and y_2 are opposite in direction; Cf. Figure 1b. Again it is easy to observe that as h approaches infinity the interface condition above may be written as

$$\left.\begin{cases} \theta = T \\[2ex] -K_{mt} \dfrac{\partial T}{\partial y_2} = K_{ft} \dfrac{\partial \theta}{\partial y_1} \end{cases}\right\} o \leqq x < \infty \quad , \quad y_1 = y_2 = o \quad , \quad t \geqq o$$

D. SPECIALIZATION OF ENERGY EQUATIONS

In what follows particularizations of equations (1) and (2) for the cylindrical geometry and equations (3) and (4) for the slab geometry are introduced, which involve various techniques for simplifying the energy equations. These methods involve radially averaging the fluid and the conduit wall temperatures by integrating with respect to radius (for cylindrical geometry) and with respect to the transverse co-ordinates (for slab geometry). Thus appropriate definitions of average temperatures and surface film heat transfer coefficient are introduced. Further simplifying assumptions are made, such as neglecting the axial conduction in the conduit wall; these simplifications give rise to interesting problems. Also since the difference between cylindrical and slab geometry disappears when both the fluid and the wall temperatures are averaged, only the cylindrical geometry is considered in this case.

1. Cylindrical Geometry

a. Radial Averaging of Temperatures

We consider isotropic conductivity in both media:

$$K_{fa} = K_{ft} = K_f \tag{5}$$

$$K_{ma} = K_{mt} = K_m \tag{6}$$

We also define the average temperatures

$$\theta^* = \frac{\int_0^a r\theta\, dr}{\int_0^a r\, dr} = \frac{2}{a^2}\int_0^a r\theta\, dr \tag{7}$$

and

$$T^* = \frac{\int_a^b rT\, dr}{\int_a^b r\, dr} = \frac{2}{b^2 - a^2}\int_a^b rT\, dr \tag{8}$$

and we introduce an "equivalent" surface film coefficient

$$h^* = \frac{-K_f\left(\frac{\partial\theta}{\partial r}\right)_{r=a}}{\theta^* - T^*} = \frac{-K_m\left(\frac{\partial T}{\partial r}\right)_{r=a}}{\theta^* - T^*} \tag{9}$$

Note that h^* relates heat flux to difference between average fluid temperature and average wall temperature.

Then, by integrating equations (1) and (2) with respect to radius, between appropriate limits, and introducing the definitions above, we find

$$\frac{\partial^2\theta^*}{\partial x^2} - \frac{V}{\alpha_f}\frac{\partial\theta^*}{\partial x} = \frac{1}{\alpha_f}\frac{\partial\theta^*}{\partial t} + \frac{h^* P}{A_f K_f}\left(\theta^* - T^*\right) \tag{10}$$

for the fluid mean temperature, and

$$\frac{\partial^2 T^*}{\partial x^2} = \frac{1}{\alpha_m}\frac{\partial T^*}{\partial t} - \frac{h^* P}{A_m K_m}\left(\theta^* - T^*\right) \tag{11}$$

for the pipe wall mean temperature. Equations (10) and (11) are essentially those considered by Hiep [13]; although his equations appear to provide for more generality (in order to apply to porous media), a change of variable shows that they are mathematically the same. The introduction of h^* may legitimately be objected to on the grounds that physically it would be quite difficult to evaluate. However, its use provides a degree of generalization compared to the physically reasonable simplifying assumption that radial variation of temperature in the fluid is so small that $\theta^* = \theta\ (r = a) = T\ (r = a)$. Moreover, even in cases where $h \to \infty$, the use of $h^* < \infty$ permits taking into account some resistance to radial heat flow in the fluid.

The initial and boundary conditions for (10) and (11) follow naturally from those of (1) and (2).

(1) <u>Initial Condition t = 0.</u>

$$\theta^* = T^* = o \qquad o \leqq x < \infty \quad , \qquad t = o$$

(2) <u>Bouhdary Condition at x = 0.</u>

$$\theta^* = T^* = 1 \qquad x = o \quad , \qquad t \geqq o$$

(3) <u>Boundary Condition as $x \to \infty$.</u>

$$\theta^* \to 0 \ , \ T^* \to o \qquad x \to \infty \quad , \qquad t \geqq o$$

b. Case where h^* Approaches Infinity

An interesting special case may be obtained if it is assumed that radial diffusion is so great that the temperature is essentially constant throughout any particular cross section. This is accomplished by letting h^* approach infinity. Then from (11) we obtain

$$\left(\theta^* - T^*\right) = \frac{A_m K_m}{h^* P}\left(\frac{1}{\alpha_m}\frac{\partial T^*}{\partial t} - \frac{\partial^2 T^*}{\partial x^{*2}}\right) \tag{12}$$

Substituting for $(\theta^* - T^*)$ from (12) above in (10), and noting that as h^* approaches infinity, then $\theta^* = T^*$, yields

$$\alpha_f\left(1 + \frac{K_m A_m}{K_f A_f}\right)\frac{\partial^2 T^*}{\partial x^2} - V\frac{\partial T^*}{\partial x} = \left(1 + \frac{\rho_m A_m C_m}{\rho_f A_f C_f}\right)\frac{\partial T^*}{\partial t} \tag{13}$$

for the common fluid and conduit wall temperature distribution. The initial and boundary conditions for (13) follow naturally from those of (10) and (11).

(1) Initial Condition, $t = 0$.

$T^* = 0 \qquad 0 \leq x < \infty \,, \qquad t = 0$

(2) Boundary Condition at $x = 0$.

$T^* = 1 \qquad x = 0 \,, \qquad t \geq 0$

(3) Boundary Condition as $x \to \infty$.

$T^* \to 0 \qquad x \to \infty \,, \qquad t \geq 0$

The solution of this system is treated in section VII.

c. Case where $K_m = 0$

By neglecting axial conduction in the solid pipe wall, a tractable system is obtained. We rewrite equation (10) with no change

$$\frac{\partial^2 \theta}{\partial x^2} - \frac{V}{\alpha_f}\frac{\partial \theta^*}{\partial x} = \frac{1}{\alpha_f}\frac{\partial \theta^*}{\partial t} + \frac{h^* P}{A_f K_f}\left(\theta^* - T^*\right) \tag{10}$$

Multiplying equation (11) by K_m and allowing K_m to approach zero, yields

$$\frac{\partial T^*}{\partial t} = \frac{h^* P}{\rho_m A_m c_m} \left(\theta^* - T^* \right) \tag{14}$$

The initial and boundary conditions are

(1) <u>Initial Condition at t = 0.</u>

$\theta^* = T^* = 0 \qquad 0 \leqq x < \infty \quad , \quad t = 0$

(2) <u>Boundary Condition at x = 0.</u>

$\theta^* = 1 \qquad x = 0 \quad , \quad t \geqq 0$

(3) <u>Boundary Condition as $x \to \infty$.</u>

$\theta^* \to 0 \qquad x \to \infty \quad , \quad t \geqq 0$

The solution of this system is dealt with in section VI .

d. Another Special Case

If radial averaging is done only for the fluid and variation with respect to radius is retained in the solid pipe wall an interesting problem results. However, this situation leads to an integro-differential equation the importance of which does not seem to be commensurate with its difficulty. Accordingly an additional simplification is made: axial conduction is neglected in the solid pipe wall. To accomplish this we set $K_{ma} = 0$ in equation (2) and we also write $K_{mt} = K_m$. For the fluid we write $K_{fa} = K_{ft} = K_f$. Note that the radial effect is absorbed in the radial averaging for the fluid. It is necessary to define a new equivalent heat transfer coefficient $\tilde{h}$ as follows.

$$-K_m \frac{\partial T}{\partial r} = -K_f \frac{\partial \theta}{\partial r} = \tilde{h}(\theta^* - T) \quad at \quad r = a \tag{15}$$

Then equations (1) and (2) with θ^* still defined by (7) reduce to

$$\frac{\partial^2 \theta^*}{\partial x^2} - \frac{V}{\alpha_f}\frac{\partial \theta^*}{\partial x} = \frac{1}{\alpha_f}\frac{\partial \theta^*}{\partial t} + \frac{\tilde{h} P}{A_f K_f}\left[\theta^* - (T)_{r=a}\right] \tag{16}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) = \frac{1}{\alpha_m}\frac{\partial T}{\partial t} \tag{17}$$

(17) is exactly the form taken by (2) with $K_{ma} = 0$. (16) is the same as 10) except for the new definition of surface film coefficient. The equivalent heat transfer coefficient $\tilde{h}$, defined by (15) relates interface heat flux to the difference between conduit <u>surface</u> temperature and fluid <u>mean</u> temperature; compare with h^* defined by (9). Clearly $\tilde{h} < h$. The reasons for introducing $\tilde{h}$ are covered in the remarks following (11).

The appropriate boundary and initial conditions are:

(1) <u>Initial Condition.</u> The initial conditions for equations (16) and (17) are exactly the same as those for equations (1) and (2)

$$\theta^* = 0 \qquad 0 \leq x < \infty \quad , \quad t = 0$$

$$T = 0 \qquad a \leq r \leq b \quad , \quad t = 0$$

(2) Boundary Condition at $x = 0$. The same conditions as stated for equations (1) still apply to equation (16):

$$\theta^* = 1 \qquad x = 0 \quad , \quad t \geqq 0$$

We observe that since equation (17) is explicitly independent of x, it is therefore not required to specify any condition at $x = 0$ for equation (17).

(3) Boundary Condition as $x \rightarrow \infty$. It is again required that θ^* remain finite; viz.

$$\theta^* = 0 \qquad x \rightarrow \infty \quad , \quad t \geqq 0$$

(4) Boundary Condition at $r = b$. Assuming perfect insulation at the outer surface of the cylinder, then

$$\frac{\partial T}{\partial r} = 0 \qquad r = b \quad , \quad t \geqq 0$$

(5) Boundary Condition at $r = a$. At the interface of fluid and pipe wall solid the continuity of heat flux requires

$$-K_m \frac{\partial T}{\partial r} = \tilde{h}\,(\theta^* - T) \qquad r = a \quad , \quad t \geqq 0$$

We note here that if $\tilde{h}$ approaches infinity, then the condition of continuity of heat flux above reduces to equality of temperature at the fluid and wall interface.

This system is analyzed in VB1. See discussion at end of IID2 for explanation of why the system of IID2 is treated first.

2. Slab Geometry

For all the simplified cases in Section IID1 in which average temperatures in both fluid and pipe wall are considered, the geometry of the situation is fully described by the parameters A_f, A_m, and P. Thus, specializing the case of slab geometry in the same ways leads, as one may easily verify, to exactly the same equations. However, the case treated in IID1d still retains radial variation in the pipe wall. Accordingly, we expect that the slab geometry counterpart will be described by slightly different equations, which we now proceed to obtain.

a. Differential Equation

Treating equations (3) and (4) for the case of flat wall duct similarly we define

$$\theta^* = \frac{\int_0^a \theta \, dy}{\int_0^a dy} = \frac{1}{a}\int_0^a \theta \, dy \tag{18}$$

Then considering isotropic conductivity in the fluid, integration of (3) yields

$$\frac{\partial^2 \theta^*}{\partial x^2} - \frac{v}{\alpha_f}\frac{\partial \theta^*}{\partial x} = \frac{1}{\alpha_f}\frac{\partial \theta^*}{\partial t} + \frac{\tilde{h}}{a K_f}\left(\theta^* - T\right) \tag{19}$$

for the fluid. Similarly with the assumption of zero axial conductivity in the wall, we obtain

$$\frac{\partial^2 T}{\partial y_2^2} = \frac{1}{\alpha_m} \frac{\partial T}{\partial t} \tag{20}$$

for the solid wall. Note the introduction of $\tilde{h}$ in (19); see remarks following (17).

b. Fluid; Initial and Boundary Conditions

(1) Initial Condition t = 0.

$\theta^* = 0 \qquad 0 \leqq x < \infty \quad , \quad t = 0$

(2) Boundary Conditions at x = 0.

$\theta^* = 1 \qquad x = 0 \quad , \quad t \geqq 0$

(3) Boundary Conditions as $x \to \infty$.

$\theta^* \longrightarrow 0 \qquad x \longrightarrow \infty \quad , \quad t \geqq 0$

c. Solid Wall; Initial and Boundary Conditions.

(1) Initial Condition, t = 0.

$T = 0 \qquad 0 \leqq y_2 \leqq b \quad , \quad t = 0$

(2) Boundary Condition at $y_2 = b$.

$\frac{\partial T}{\partial y_2} = 0 \qquad y_2 = b \quad , \quad t \geqq 0$

(3) Boundary Condition at $y_2 = 0$.

$-K_m \frac{\partial T}{\partial y_2} = \tilde{h}\,(\theta^* - T) \quad y_2 = 0 \quad , \quad t \geqq 0$

This system which is similar to that obtained in subsection IID1d is somewhat easier to deal with. Accordingly, it is treated first, in VA while that for the system IID1d is dealt with in VB.

III. INTEGRAL EQUATION FORMULATION

A. INTRODUCTION AND PLAN OF ANALYSIS

Equations (II-3) and (II-4), with $K_{fa} = K_{ft} = K_f$ and $K_{ma} = K_{mt} = K_m$ so as to provide isotropic thermal conductivity, involve two dependent variables, $\theta(x, y, t)$ and $T(x, y, t)$, each involving three independent variables, along with a formidable number of boundary and initial conditions. Similar remarks hold for the case of cylindrical geometry. These are complicated systems to deal with and we have achieved solutions which are useful for calculating purposes only by making various simplifying assumptions. These solutions are given in other sections of this thesis. In this section we will represent the system of equations (II-3) and (II-4) in another form which involves only one unknown function of two independent variables. No additional conditions are required for the complete specification of this unknown function. The form of this equation, (70), is that of an integral equation of what is known as "first type" (that is, the unknown function does not appear outside an integral sign) with complicated kernels and forcing functions.

Although the method of procedure is essentially the same for the case of cylindrical geometry as for the case of slab geometry, the details are sufficiently different to justify treating the two cases separately. We treat the case of slab geometry first and in subsection IIID we return to the case of cylindrical geometry. In each case the method of procedure is essentially as described in the next paragraph.

We define an unknown function $\phi(x, t)$ which represents the (unknown) temperature of the fluid at the interface and a similar function $\psi(x, t)$ which represents the unknown temperature of the solid at the interface. For the fluid we solve the problem of finding the interior temperature $\theta(x, y, t)$ when the surface temperature is given by $\phi(x, t)$, subject to other appropriate boundary and initial conditions. It is convenient to consider this solution as the sum of two solutions, one involving a homogeneous boundary condition at the interface and accommodating the boundary condition at the end, $x = 0$, and a second solution involving a homogeneous boundary condition for $x = 0$ and the unknown function $\phi(x, t)$ at the interface. The solution to the latter problem is expressed in terms of an integral involving the function $\phi(x, t)$; it is obtained by use of a procedure which is due to A. N. Lowan [20, 21]. We proceed in the same way to express the solution for the solid temperature $T(x, y, t)$ in terms of the unknown surface temperature $\psi(x, t)$. The interface heat balance condition provides two relations, one of which may be used to express the function $\psi(x, t)$ in terms of the function $\phi(x, t)$. The other relation then becomes the integral equation for the unknown function $\phi(x, t)$.

B. AUXILIARY PROBLEMS (SLAB GEOMETRY)

We will deal with the problem of the temperature distribution in the fluid first. Then, by taking the flow velocity as zero and by using subscript m (for the solid) rather than f (for the fluid), and

by changing the designation of the surface temperature distribution from $\phi(x,t)$ to $\psi(x,t)$, we can very quickly obtain the corresponding solution for the temperature distribution in the solid. We immediately write

$$\theta(x,y,t) = u_1(x,y,t) + u_2(x,y,t) \tag{1}$$

where $u_i(x,y,t)$ are the solutions to the two auxiliary problems considered in the remainder of this subsection.

1. <u>Fluid; Homogeneous Boundary Condition at Interface</u>

For simplicity in this subsection we will omit subscripts, and will replace them when we combine solutions: α will denote α_f, u will denote u_1, y will denote y_1 (Cf. Figure 1b). Thus the problem of interest is

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} - \frac{V}{\alpha}\frac{\partial u}{\partial x} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \tag{2}$$

The boundary and initial conditions are

$$u = 1 \qquad x = 0 \qquad 0 \leq y \leq a \qquad t \geq 0 \tag{3}$$

$$u \to 0 \qquad x \to \infty \qquad 0 \leq y \leq a \qquad t \geq 0 \tag{4}$$

$$u = 0 \qquad 0 \leq x < \infty \qquad y = 0 \qquad t \geq 0 \tag{5}$$

$$\frac{\partial u}{\partial y} = 0 \qquad 0 \leq x < \infty \qquad y = a \qquad t \geq 0 \tag{6}$$

$$u = 0 \qquad 0 \leq x < \infty \qquad 0 \leq y \leq a \qquad t = 0 \tag{7}$$

Using an overbar to denote Laplace transform with respect to time, we obtain

$$\frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial x^2} - \frac{V}{\alpha} \frac{\partial \bar{u}}{\partial x} = \frac{s}{\alpha} \bar{u} \tag{8}$$

This incorporates the initial condition. All the boundary conditions will be the same in Laplace transform form except that at $x = 0$ which becomes $\bar{u} = \frac{1}{s}$. Thus assuming a solution of the form

$$\bar{u} = A\, e^{-\lambda x} \sin(\mu y) \tag{9}$$

we observe that all conditions are satisfied except that for $x = 0$, and the condition of zero slope at $y = a$. Imposing the latter yields the characteristic values

$$\mu_n = (n+1/2)\pi/a \qquad n = 0, 1, 2, \ldots \tag{10}$$

Because of linearity we can write

$$\bar{u}(x, y, s) = \sum_{n=0}^{\infty} A_n\, e^{-\lambda_n x} \sin(\mu_n y) \tag{11}$$

Substitution of (11) into (8) yields

$$\lambda_n = -\frac{V}{2\alpha} \pm \sqrt{\left(\frac{V}{2\alpha}\right)^2 + \mu_n^2 + \frac{s}{\alpha}} \tag{12}$$

The negative root is discarded in order to assure boundedness as $x \longrightarrow \infty$. The condition at $x = 0$ and use of the orthogonality property of trignometric functions yields

$$A_n = \frac{1}{s} \quad \frac{2}{\mu_n a} \tag{13}$$

Substituting from (13) and (12) into (11) we obtain

$$\bar{u}(x,y,s) = \sum_{n=0}^{\infty} \frac{2}{\mu_n a} \sin(\mu_n y) \exp\left[\frac{Vx}{2\alpha} - x\sqrt{\mu_n^{*2} + \frac{s}{\alpha}}\right] \tag{14}$$

where

$$\mu_n^{*2} = \mu_n^2 + \left(\frac{V}{2\alpha}\right)^2 \tag{15}$$

The inversion of (14) may be obtained using standard tables [29]. Thus

$$u(x,y,t) = \sum_{n=0}^{\infty} Y_1(y)\, X_1(x,t) \tag{16}$$

where

$$Y_1(y) = \frac{\sin \mu_n y}{\mu_n} \tag{17}$$

$$X_1(x,t) = \frac{1}{a} e^{\frac{Vx}{2\alpha}} \left[e^{-\mu_n^* x} \cdot \operatorname{erfc}\left(\frac{x - 2\alpha\mu_n^* t}{2\sqrt{\alpha t}}\right) + e^{\mu_n^* x} \cdot \operatorname{erfc}\left(\frac{x + 2\alpha\mu_n^* t}{2\sqrt{\alpha t}}\right)\right] \tag{18}$$

In these expressions, although there is explicit dependence upon the integer n, this is not reflected in the notation on the left. This is done merely for simplicity of notation and will also be done with similar definitions to follow. In every case the symbols are used under a sign of summation with respect to n.

For future purposes we evaluate the partial derivative of (16) with respect to y and evaluate at y = 0.

$$\left[\frac{\partial u}{\partial y}\right]_{y=0} = \sum_{n=0}^{\infty} X_1(x,t) \tag{19}$$

where

$$Y_1'(0) = 1 \tag{20}$$

As the reader may come to appreciate, when the many preliminary results are assembled into the final equation, a compact notation is convenient. Functions Y_1 and X_1 and constants μ_n^* are the first of several such notations which will be introduced.

2. <u>Fluid; Non-homogeneous Boundary Condition at Interface</u>

In what follows u stands for $u_2(x, y, t)$ and it is governed by the same equation (2). However, the following changes are made in the boundary conditions (3) and (5)

$$u = 0 \quad \text{(rather than } u = 1) \quad x = 0 \quad 0 \leqq y \leqq a \quad t \geqq 0$$

$$u = \phi(x, t) \quad \text{(rather than } u = 0) \quad 0 \leqq x \leqq \infty \quad y=0 \quad t \geqq 0$$

where $|\phi(x, t)| \leqq M < \infty$ for all x and t. We further make the following substitution in order to drop out the convection term.

$$u(x, y, t) = W(x, y, t) \exp\left[\frac{Vx}{2\alpha} - \frac{V^2 t}{4\alpha}\right] \tag{21}$$

Then equation (2) and the appropriate boundary and initial conditions reduce to the following equation and conditions for W(x, y, t).

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = \frac{1}{\alpha}\frac{\partial W}{\partial t} \tag{22}$$

$$W = 0 \qquad x = 0 \quad , \quad 0 \leqq y \leqq a \quad , \quad t \geqq 0 \tag{23}$$

$$W < \infty \qquad x \to \infty \quad , \quad 0 \leqq y \leqq a \quad , \quad t \geqq 0 \tag{24}$$

$$W = g(x,t) \qquad 0 \leqq x < \infty \quad , \quad y = 0 \quad , \quad t \geqq 0 \tag{25}$$

$$\frac{\partial W}{\partial y} = 0 \qquad 0 \leqq x < \infty \quad , \quad y = a \quad , \quad t \geqq 0 \tag{26}$$

$$W = 0 \qquad 0 \leqq x < \infty \quad , \quad 0 \leqq y \leqq a \quad , \quad t = 0 \tag{27}$$

where

$$g(x,t) = \phi(x,t) \exp\left[\frac{-Vx}{2\alpha} + \frac{V^2 t}{4\alpha}\right] \tag{28}$$

Again taking Laplace transformations with respect to t and accounting for the initial condition we obtain

$$\frac{\partial^2 \overline{W}}{\partial x^2} + \frac{\partial^2 \overline{W}}{\partial y^2} = \frac{s}{\alpha}\overline{W} \tag{29}$$

All the boundary conditions will be exactly the same as above except (25), which becomes $\overline{W} = \overline{g}(x, s)$.

Following the method of solution used by Lowan [20, 21] we consider the function

$$\bar{W}(x,y,s) = \frac{2}{\pi}\int_0^\infty\int_0^\infty \bar{F}(\alpha,y;\lambda)\,\bar{g}(\xi,s)\sin\mu x\,\sin\mu\xi\,d\mu\,d\xi \tag{30}$$

where

$$\bar{F}(\alpha,y;\lambda) = \frac{\cosh[\lambda(\alpha-y)]}{\cosh(\lambda\alpha)} \tag{31}$$

and

$$\lambda^2 = \mu^2 + \frac{s}{\alpha} \tag{32}$$

Note that μ and ξ are dummy variables of integration. Direct substitution shows that (29) is satisfied. It is easy to verify that all the boundary and initial conditions are satisfied. In particular we note that

$$\bar{W}(x,0,s) = \frac{2}{\pi}\int_0^\infty\int_0^\infty \bar{g}(\xi,s)\sin\mu\xi\,\sin\mu x\,d\mu\,d\xi = \bar{g}(x,s)$$

by the Fourier identity. As regards boundedness

$$\left|\bar{W}(x,y,s)\right| \leqq \frac{2M}{\pi}\int_0^\infty\int_0^\infty \sin\mu x\,\sin\mu\xi\,d\mu\,d\xi = \frac{2M}{\pi} < \infty$$

This completes the proof that (30) is the desired solution of (29) which satisfies all the auxiliary conditions.

Continuing to follow the procedure described by Lowan [20, 21], we define

$$\int_0^{\infty} e^{-st} F_\mu(a, y, t)\, dt = \bar{F}(a, y; \lambda) = \bar{F}\left(a, y; \sqrt{\mu^2 + \frac{s}{\alpha}}\right) \tag{33}$$

i.e., F_μ (a, y;t) is the inverse of $\overline{F}$ (a, y; λ). The subscript μ indicates the dependence upon the parameter μ ; the dependence of $\overline{F}$ upon s is made clear by expressing λ in terms of s, as above. The convolution theorem then gives

$$W(x, y, t) = \frac{2}{\pi} \int_0^{\infty}\int_0^{\infty} \sin \mu x \sin \mu \xi \, d\mu \, d\xi \int_0^{t} F_\mu(a, y; \eta)\, g(\xi, t-\eta)\, d\eta \tag{34}$$

Further defining

$$G_\mu(a, y, t) = \int_0^{t} F_\mu(a, y, t)\, dt \tag{35}$$

then

$$\frac{\partial G_\mu(a, y, t)}{\partial t} = F_\mu(a, y, t) \tag{36}$$

Thus substituting from (36) in (34) yields

$$W(x,y,t) = \frac{2}{\pi}\int_0^\infty\int_0^\infty \sin\mu x \sin\mu\xi \, d\mu \, d\xi \int_0^t \frac{\partial G_\mu(a,y;\eta)}{\partial \eta} g(\xi, t-\eta) d\eta \quad (37)$$

If we now take the Laplace transform of (35), we obtain

$$\mathcal{L}\left[G_\mu(a,y;t)\right] = \frac{1}{s}\mathcal{L}\left[F_\mu(a,y;t)\right] = \frac{1}{s}\bar{F}(a,y;\lambda) \quad (38)$$

$$= \frac{1}{s}\frac{Y(s)}{Z(s)}$$

where

$$Y(s) = \cosh\left[\lambda(a-y)\right] \quad ; \quad Z(s) = \cosh(\lambda a) \quad (39)$$

The singularities of (38) are simple poles corresponding to the zeros of cosh(λ a); of course, these are functions of the variable μ . Note that $\cosh(\lambda a) = 1 + \lambda^2 a^2/2! + \lambda^4 a^4/4! + \ldots$ which converges everywhere in the finite plane. Thus, although λ itself involves a square root , λ^2 does not, and there is no difficulty to be encountered dealing with branches. Thus, the residue theorem affords the desired inverse of (38)

$$G_\mu(a,y;t) = \frac{Y(0)}{Z(0)} + \sum_{n=0}^{\infty} \frac{Y(s_n)\, e^{s_n t}}{s_n\left[\frac{\partial Z}{\partial s}\right]_{s=s_n}} \quad (40)$$

The summation in (40) extends over the roots of

$$\cosh(\lambda a) = 0 \tag{41}$$

namely the numbers

$$\lambda_n = \sqrt{\mu^2 + \frac{S_n}{\alpha}} = (n + 1/2)\frac{i\pi}{a} = i\,\mu_n \qquad n = 0, 1, 2, \ldots \tag{42}$$

(Do not confuse the constants μ_n of (10) with the continuous variable μ of (30)). Thus

$$\begin{aligned} S_n &= -\alpha\left[(n+1/2)^2 \frac{\pi^2}{a^2} + \mu^2\right] \\ &= -\alpha\left(\mu_n^2 + \mu^2\right) \end{aligned} \tag{43}$$

where S_n are the poles of (38) in excess of the simple pole at $S = 0$. Note that they are functions of the variable μ. Therefore the formal inverse of (38) is

$$G_\mu(a, y; t) = \frac{\cosh[\mu(a-y)]}{\cosh(\mu a)} - \frac{2\pi}{a^2}\sum_{n=0}^{\infty} \frac{(-1)^n (n+1/2)\cos[\mu_n(a-y)]\, e^{-\alpha(\mu_n^2+\mu^2)t}}{(\mu_n^2 + \mu^2)} \tag{44}$$

The following derivative is needed in (37)

$$\frac{\partial G_\mu(a, y; t)}{\partial t} = \frac{2\alpha}{a}\sum_{n=0}^{\infty} (-1)^n \mu_n \cos\left[\mu_n(a-y)\right] e^{-\alpha(\mu_n^2+\mu^2)t} \tag{45}$$

Hence, substituting from (45) in (37) we obtain

$$W(x,y,t) = \frac{4\alpha}{\pi a}\sum_{n=0}^{\infty}(-1)^n \mu_n \cos\left[\mu_n(a-y)\right]\int_0^{\infty}\int_0^{t} e^{-\alpha\mu_n^2\eta} g(\xi,t-\eta)\cdot \left[\int_0^{\infty} e^{-\alpha\mu^2\eta} \sin\mu x \sin\mu\xi \, d\mu\right] d\eta\, d\xi \tag{46}$$

In order to complete the analysis, substitute (28) in (46) and then substitute the result into (21). Thus we obtain

$$u(x,y,t) = \frac{4\alpha}{\pi a}\sum_{n=0}^{\infty}(-1)^n \mu_n \cos\left[\mu_n(a-y)\right]\int_0^{\infty}\int_0^{t} e^{\frac{V}{2\alpha}(x-\xi)}\cdot e^{-\eta\left(\frac{V^2}{4\alpha}+\alpha\mu_n^2\right)} \cdot \phi(\xi,t-\eta)\left[\int_0^{\infty} e^{-\alpha\mu^2\eta} \sin\mu x \sin\mu\xi \, d\mu\right] d\eta\, d\xi \tag{47}$$

We are able to carry out the integration with respect to μ so that we finally obtain

$$u(x,y,t) = \sum_{n=0}^{\infty} Y_2(y)\int_0^{\infty}\int_0^{t} \phi(\xi,t-\eta)\, K_1(\xi,x,t,\eta)\, d\eta\, d\xi \tag{48}$$

where

$$Y_2(y) = \frac{1}{\mu_n}(-1)^n \cos\left[\mu_n(a-y)\right] \tag{49}$$

$$K_1(x,\xi,t,\eta) = \sqrt{\frac{\alpha}{\pi}}\,\frac{\mu_n^2}{\alpha}\, e^{\frac{V}{2\alpha}(x-\xi)}\cdot e^{-\eta\left(\frac{V^2}{4\alpha}+\alpha\mu_n^2\right)} \cdot \frac{1}{\sqrt{\eta}}\left\{\exp\left[\frac{-(\xi-x)^2}{4\alpha\eta}\right] - \exp\left[\frac{-(\xi+x)^2}{4\alpha\eta}\right]\right\} \tag{50}$$

For later purposes we need $\left[\frac{\partial u}{\partial y}\right]_{y=0}$ which is evaluated below.

$$\left[\frac{\partial u}{\partial y}\right]_{y=0} = \sum_{n=0}^{\infty} \int_0^{\infty}\int_0^{t} \phi(\xi, t-\eta)\, K_1(x, \xi, t, \eta)\, d\eta\, d\xi \tag{51}$$

In order to assure that there is no misinterpretation, we remark that the integration with respect to ξ is from zero to infinity and that with respect to η extends from zero to t.

3. <u>Combination of Solutions for the Fluid</u>

Simple addition of (16) and (48) yields

$$\theta(x, y_1, t) = \sum_{n=0}^{\infty}\left[Y_1(y_1)X_1(x,t) + Y_2(y_1)\int_0^{\infty}\int_0^{t}\phi(\xi, t-\eta)K_1(x,\xi,t,\eta)\,d\eta\,d\xi\right] \tag{52}$$

Likewise by superimposing (19) and (51) we obtain,

$$\left[\frac{\partial \theta}{\partial y_1}\right]_{y_1=0} = \sum_{n=0}^{\infty}\left[X_1(x,t) + \int_0^{\infty}\int_0^{t}\phi(\xi, t-\eta)\, K_1(x,\xi,t,\eta)\,d\eta\,d\xi\right] \tag{53}$$

where, having restored the appropriate subscripts,

$$Y_1(y_1) = \frac{\sin \mu_n y_1}{\mu_n} \tag{54}$$

$$X_1(x,t) = \frac{1}{a}\, e^{\frac{Vx}{2\alpha_f}}\left[e^{-\mu_n^* x}\cdot \operatorname{erfc}\left(\frac{x-2\alpha_f\mu_n^* t}{2\sqrt{\alpha_f t}}\right) + e^{\mu_n^* x}\cdot \operatorname{erfc}\left(\frac{x+2\alpha_f\mu_n^* t}{2\sqrt{\alpha_f t}}\right)\right] \tag{55}$$

$$Y_2(y_1) = \frac{(-1)^n}{\mu_n}\cos\left[\mu_n(a-y)\right] \tag{56}$$

$$K_1(x,\xi,t,\eta) = \sqrt{\frac{\alpha_f}{\pi}}\,\frac{\mu_n^2}{\alpha}\cdot e^{\frac{V}{2\alpha_f}(x-\xi)}\cdot e^{-\eta\left(\frac{V}{2\alpha_f}+\alpha_f\mu_n^2\right)}$$

$$\cdot\frac{1}{\sqrt{\eta}}\left\{\exp\left[\frac{-(\xi-x)^2}{4\alpha_f\eta}\right]-\exp\left[\frac{-(\xi+x)^2}{4\alpha_f\eta}\right]\right\} \tag{57}$$

Also, of course, (10) and (15) define μ_n and μ_n^*.

It is well to pause at this point to review the physical nature of the problem for which (52) provides the solution. This is given below. The differential equation is

$$\frac{\partial^2\theta}{\partial x^2} - \frac{V}{\alpha_f}\frac{\partial\theta}{\partial x} + \frac{\partial^2\theta}{\partial y_1^2} = \frac{1}{\alpha_f}\frac{\partial\theta}{\partial t} \tag{II-3}$$

The auxiliary conditions are

$\theta = 1$	$x = 0$,	$0 \leq y_1 \leq a$,	$t \geq 0$
$\theta < \infty$	$x \to \infty$,	$0 \leq y_1 \leq a$,	$t \geq 0$
$\theta = \phi(x,t)$	$0 \leq x < \infty$,	$y_1 = 0$,	$t \geq 0$
$\frac{\partial\theta}{\partial y} = 0$	$0 \leq x < \infty$,	$y_1 = a$,	$t \geq 0$
$\theta = 0$	$0 \leq x < \infty$,	$0 \leq y_1 \leq a$,	$t = 0$

4. Solution for the Solid Wall

The solution for the similar problem for the solid wall is easily obtained from the preceding solution for the fluid. All that is necessary to dois to write T in place of θ, ψ in place of ϕ, replace subscript f by subscript m, replace a by b, take V = 0, and introduce a new symbol λ_n (defined below) in place of μ_n.

$$T(x,y_2,t) = \sum_{n=0}^{\infty} \left[Y_3(y_2)\, X_2(x,t) + Y_4(y_2) \int_0^{\infty}\int_0^{t} \psi(\xi, t-\eta) K_2(x,\xi,t,\eta)\, d\eta\, d\xi \right] \quad (58)$$

where

$$Y_3(y_2) = \frac{\sin \lambda_n y_2}{\lambda_n} \quad (59)$$

$$X_2(x,t) = \frac{1}{b} \left[e^{-\lambda_n x} \cdot \operatorname{erfc}\left(\frac{x - 2\alpha_m \lambda_n t}{2\sqrt{\alpha_m t}} \right) + e^{\lambda_n x} \cdot \operatorname{erfc}\left(\frac{x + 2\alpha_m \lambda_n t}{2\sqrt{\alpha_m t}} \right) \right] \quad (60)$$

$$Y_4(y_2) = \frac{(-1)^n \cos\left[\lambda_n (b - y)\right]}{\lambda_n} \quad (61)$$

$$K_2(x,\xi,t,\eta) = \frac{\lambda_n^2}{b} \sqrt{\frac{\alpha_m}{\pi}}\; e^{-\alpha_m \lambda_n^2 \eta} \cdot \frac{1}{\sqrt{\eta}} \left\{ \exp\left[\frac{-(\xi - x)^2}{4\alpha_m \eta} \right] - \exp\left[\frac{-(\xi + x)^2}{4\alpha_m \eta} \right] \right\} \quad (62)$$

$$\lambda_n = (n + 1/2)\frac{\pi}{b} \quad (63)$$

(Do not confuse λ_n above with λ_n used temporarily in (42))

$$\left[\frac{\partial T}{\partial y_2} \right]_{y_2 = 0} = \sum_{n=0}^{\infty} \left[X_2(x,t) + \int_0^{\infty}\int_0^{t} \psi(\xi, t-\eta)\, K_2(x,\xi,t,\eta)\, d\eta\, d\xi \right] \quad (64)$$

C. DERIVATION OF INTEGRAL EQUATION (SLAB GEOMETRY)

Equations (52) and (58) thereby give the fluid and pipe wall temperature distributions respectively. However these solutions are in terms of the arbitrary interface temperature functions $\phi(x, t)$ and $\psi(x, t)$. We now return to the actual boundary conditions that prevail at the interface, viz.

$$K_f \frac{\partial \theta}{\partial y_1} = h(\phi - \psi) \qquad y_1 = 0 \quad , \quad 0 \leq x < \infty \quad , \quad t \geq 0 \tag{65}$$

and

$$-K_m \frac{\partial T}{\partial y_2} = h(\phi - \psi) \qquad y_2 = 0 \quad , \quad 0 \leq x < \infty \quad , \quad t \geq 0 \tag{66}$$

(Note: The signs on the left hand side of (65) and (66) are consistent with the choice of coordinates used in Figure 1b). Solving for ψ from (65) in terms of ϕ, we obtain

$$\psi = \phi - \frac{K_f}{h} \left(\frac{\partial \theta}{\partial y_1}\right)_{y_1 = 0} \tag{67}$$

Substitution from (53) in (67) yields

$$\psi(x,t) = \phi(x,t) - \frac{K_f}{h} \left\{ \sum_{n=0}^{\infty} \left[X_1(x,t) + \int_0^{\infty}\int_0^{t} \phi(\xi, t-\eta) \cdot K_1(x, \xi, t, \eta)\, d\eta\, d\xi \right] \right\} \tag{68}$$

Furthermore, substitution of (68) in (66) yields

$$-K_m\left[\frac{\partial T}{\partial y_2}\right]_{y_2=0} = K_f\left\{\sum_{n=0}^{\infty}\left[X_1(x,t) + \int_0^{\infty}\int_0^{t} \phi(\xi,t-\eta)K_1(x,\xi,t,\eta)\,d\eta\, d\xi\right]\right\} \quad (69)$$

Substitution of ψ (x, t) from (68) into (64) and the result into (69) yields

$$K_m\sum_{n=0}^{\infty}\left\{X_2(x,t) + \int_0^{\infty}\int_0^{t} K_2(x,\xi,t,\eta)\cdot\left[\phi(\xi,t-\eta) - \frac{K_f}{h}\sum_{m=0}^{\infty}\left(X_1(\xi,t-\eta)\right.\right.\right.$$
$$\left.\left.\left. + \int_0^{\infty}\int_0^{t-\eta}\phi(\alpha,t-\eta-\beta)\,K_1(\xi,\alpha,t-\eta,\beta)\,d\beta\, d\alpha\right)\right]\right\}d\eta\, d\xi \; + \quad (70)$$
$$K_f\sum_{n=0}^{\infty}\left[X_1(x,t) + \int_0^{\infty}\int_0^{t}\phi(\xi,t-\eta)\,K_1(x,\xi,t,\eta)\,d\eta\, d\xi\right] = 0$$

It should be clear that we could, with equal facility, express the problem in terms of ψ (x, t) instead of ϕ (x, t) as above. The distinction is trivial.

Equation (70) is the desired integral equation, the only unknown in which is the function ϕ (x, t) which represents the surface temperature of the fluid.

Equation (70) is considerably simplified if there is no surface resistance. Letting h, the heat transfer coefficient, approach infinity, reduces (70) to

$$0 = \sum_{n=0}^{\infty} \left\{ X_1(x,t) + \frac{K_m}{K_f} X_2(x,t) + \int_0^{\infty}\int_0^{t} \phi(\xi, t-\eta) \cdot \right.$$

$$\left. \left[K_1(x,\xi,t,\eta) + \frac{K_m}{K_f} K_2(x,\xi,t,\eta)\right] d\eta\, d\xi \right\} \tag{71}$$

for $h = \infty$.

D. AUXILIARY PROBLEMS (CYLINDRICAL GEOMETRY)

For the case of cylindrical geometry, the treatment is quite similar to that which has preceded. However, it is not possible to derive the equations for the pipe wall by simple manipulation of the equation for the fluid as was done above. Thus it is appropriate to consider four auxiliary problems similar to those dealt with before, two for the fluid, arriving at equation (165), and two for the solid, arriving at equation (170). These equations will then finally be combined so as to arrive at the desired integral equation. By suitably redefining the several functions which are involved, it will be possible to express this equation, (181) exactly in the form of (70).

1. <u>Fluid; Homogeneous Boundary Condition at Interface</u>

Consider the problem mathematically stated as

$$\frac{\partial^2 u}{\partial x^2} - \frac{V}{\alpha}\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \tag{72}$$

with the boundary and initial conditions:

$$u = 1 \qquad x=0 \quad , \quad 0 \leqq r \leqq a \quad , \quad t \geqq 0 \tag{73}$$

$$u = 0 \qquad x \to \infty \quad , \quad 0 \leqq r \leqq a \quad , \quad t \geqq 0 \tag{74}$$

$$\frac{\partial u}{\partial r} = 0 \qquad 0 \leqq x < \infty \quad , \quad r = 0 \quad , \quad t \geqq 0 \tag{75}$$

$$u = 0 \qquad 0 \leqq x < \infty \quad , \quad r = a \quad , \quad t \geqq 0 \tag{76}$$

$$u = 0 \qquad 0 \leqq x < \infty \quad , \quad 0 \leqq r \leqq a \quad , \quad t = 0 \tag{77}$$

Assuming a solution of the form

$$u = A J_0(\beta r) X_1(x,t) \tag{78}$$

and substituting (78) in (72) yields

$$\frac{\partial^2 X_1}{\partial x^2} - \frac{V}{\alpha} \frac{\partial X_1}{\partial x} - \beta^2 X_1 = \frac{1}{\alpha} \frac{\partial X_1}{\partial t} \tag{79}$$

Equation (78) already satisfies condition (75). Condition (76) yields the characteristic equation

$$J_0(\beta a) = 0 \tag{80}$$

the roots of which are denoted by β_n, n=1,2,... . Because of the linearity we can write

$$u = \sum_{n=1}^{\infty} A_n J_0(\beta_n r) X_1(x,t) \tag{81}$$

The solution of (79) is obtained by taking Laplace transform with respect to time, which yields a second order homogeneous ordinary differential equation the solution of which is

$$\bar{X}_1(x,t) = B \exp\left[\frac{Vx}{2\alpha} - x\sqrt{\left(\frac{V}{2\alpha}\right)^2 + \beta_n^2 + \frac{s}{\alpha}}\right] \tag{82}$$

In (82) the exponential with positive exponent has been discarded so that condition (74) may be satisfied. Substitution of (82) in (81) and absorbing B into A_n, we obtain

$$\bar{u} = \sum_{n=1}^{\infty} A_n J_0(\beta_n r) \exp\left[\frac{Vx}{2\alpha} - x\sqrt{\left(\frac{V}{2\alpha}\right)^2 + \beta_n^2 + \frac{s}{\alpha}}\right] \tag{83}$$

Thus at this point we observe that the inverse of (83) satisfies (72), (74), (75), (76) and (77). Condition (73) provides a means of evaluating A_n. Taking x = 0 (73) gives

$$\frac{1}{s} = \sum_{n=1}^{\infty} A_n J_0(\beta_n r) \tag{84}$$

multiplying by $rJ_0(\beta_m r)dr$ and using the orthogonality relation of Bessel functions;

$$\int_0^a J_0(\beta_m r) J_0(\beta_n r) r\,dr = \frac{a^2}{2} J_1(\beta_m a) \tag{85}$$

if n=m and zero if $n \neq m$, one easily finds

$$A_n = \frac{1}{s} \frac{2}{\beta_n a} \frac{1}{J_1(\beta_n a)} \tag{86}$$

Thus we now have

$$\bar{u} = \sum_{n=1}^{\infty} \frac{2}{\beta_n a} \frac{J_0(\beta_n r)}{J_1(\beta_n a)} \left\{ \frac{1}{s} \exp\left[\frac{Vx}{2\alpha} - x\sqrt{\left(\frac{V}{2\alpha}\right)^2 + \beta_n^2 + \frac{s}{\alpha}}\right]\right\} \tag{87}$$

The inversion of (87) is tabulated [29]; viz.

$$u(r,x,t) = \sum_{n=1}^{\infty} R_1(r)\ X_1(x,t) \tag{88}$$

where

$$R_1(r) = \frac{1}{\beta_n} \cdot \frac{J_0(\beta_n r)}{J_1(\beta_n a)} \tag{89}$$

$$X_1(x,t) = \frac{1}{a} e^{\frac{Vx}{2\alpha}} \left[e^{-\omega_n x} \cdot \mathrm{erfc}\left(\frac{x - 2\omega_n \alpha t}{2\sqrt{\alpha t}}\right) + e^{\omega_n x} \cdot \mathrm{erfc}\left(\frac{x + 2\omega_n \alpha t}{2\sqrt{\alpha t}}\right) \right] \quad 90)$$

$$\omega_n^2 = \beta_n^2 + \left(V/2\alpha\right)^2$$

For future purposes we need the partial derivative of (88) with respect to radius evaluated at r = a.

$$\left[\frac{\partial u}{\partial r}\right]_{r=a} = -\sum_{n=1}^{\infty} X_1(x,t) \tag{91}$$

2. Fluid; Non-homogeneous B.C. at Interface

The problem considered here is the same as that in the preceding subsection except that the following changes are made to the boundary conditions (73) and (76).

$u = 0$ (rather than $u = 1$) $\quad x = 0, \quad 0 \leq r \leq a, \quad t \geq 0$

$u = \phi(x,t)$ (rather than $u = 0$) $\quad 0 \leq x < \infty, \quad r = a, \quad t \geq 0$

where $|\phi(x,t)| \leq M < \infty$ for all x, t. A substitution of the form

$$u(r,x,t) = W(r,x,t)\, \exp\left(\frac{Vx}{2\alpha} - \frac{V^2 t}{4\alpha}\right) \tag{92}$$

reduces (72) to

$$\frac{\partial^2 W}{\partial x^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{\partial^2 W}{\partial r^2} = \frac{1}{\alpha}\frac{\partial W}{\partial t} \tag{93}$$

and all the boundary and initial conditions will remain the same except that at r = a we have

$$W(a, x, t) = \phi(x, t) \exp\left(\frac{-Vx}{2\alpha} + \frac{V^2 t}{4\alpha}\right) = g(x, t) \tag{94}$$

Taking Laplace transform of (93), we obtain

$$\frac{\partial^2 \bar{W}}{\partial x^2} + \frac{1}{r}\frac{\partial \bar{W}}{\partial r} + \frac{\partial^2 \bar{W}}{\partial r^2} = \frac{s}{\alpha}\bar{W} \tag{95}$$

Again all the boundary conditions will be the same except (94) which in Laplace transforms will become

$$\bar{W}(a, x, s) = \bar{g}(x, s) \qquad 0 \leqq x < \infty \qquad r = a \tag{96}$$

Following the method of solution used by Lowan we consider the function

$$\bar{W}(r, x, s) = \frac{2}{\pi}\int_0^\infty\int_0^\infty \bar{F}(a, r; \lambda)\, \bar{g}(\xi, s) \sin \mu x \sin \mu\xi \, d\mu \, d\xi \tag{97}$$

where

$$\bar{F}(a, r; \lambda) = \frac{I_0(\lambda r)}{I_0(\lambda a)} \tag{98}$$

and

$$\lambda^2 = \mu^2 + \frac{s}{\alpha} \tag{99}$$

Note that μ and ξ are dummy variables of integration. Direct substitution shows that (95) is satisfied as well as all the boundary conditions. In particular once again we note that

$$\begin{aligned} \bar{W}(a,x,s) &= \frac{2}{\pi}\int_0^\infty\int_0^\infty \bar{g}(\xi,s)\sin\mu x \sin\mu\xi \, d\mu\, d\xi \\ &= \bar{g}(x,s) \end{aligned} \tag{100}$$

by the Fourier identity. As regards boundedness,

$$\left|\bar{W}(r,x,s)\right| \leqq \frac{2M}{\pi}\int_0^\infty\int_0^\infty \sin\mu x \sin\mu\xi \, d\mu\, d\xi = \frac{2M}{\pi} < \infty \tag{101}$$

This completes the proof that (97) is a solution of (95) which satisfies all auxiliary conditions. Let us now define

$$\int_0^\infty e^{-st} F_\mu(a,r;t)dt = \bar{F}(a,r;\lambda) = \bar{F}\left(a,r;\sqrt{\mu^2+\frac{s}{\alpha}}\right) \tag{102}$$

i.e., F_μ (a, r;t) is the inverse of $\overline{F}$ (a, r; λ). Note again that the subscript μ indicates the dependence upon the parameter μ ; the dependence of $\overline{F}$ upon s is clearly indicated by expressing λ in terms of s, as shown in (102). The convolution theorem then gives

$$W(r,x,t) = \frac{2}{\pi}\int_0^\infty\int_0^\infty \sin\mu x \sin\mu\xi \, d\mu\, d\xi \int_0^t F_\mu(a,r;\eta)\, g(\xi,t-\eta)\, d\eta \tag{103}$$

Further, defining

$$G_\mu(a,r;t) = \int_0^t F_\mu(a,r;t)dt \tag{104}$$

gives

$$\frac{\partial G_\mu(a,r;t)}{\partial t} = F_\mu(a,r;t) \tag{105}$$

Substituting from (105) in (103) yields

$$W(r,x,t) = \frac{2}{\pi}\int_0^\infty\int_0^\infty \sin\mu x \sin\mu\xi \, d\mu d\xi \int_0^t \frac{\partial}{\partial\eta} G_\mu(a,r;\eta)\, g(\xi,t-\eta)d\eta \tag{106}$$

If we now take the Laplace transform of (104), we obtain

$$\begin{aligned} \mathcal{L}\left[G_\mu(a,r;t)\right] &= \int_0^\infty e^{-st} G_\mu(a,r;t)\,dt \\ &= \frac{1}{s}\mathcal{L}\left[F_\mu(a,r;t)\right] \\ &= \frac{1}{s}\bar{F}(a,r;\lambda) \\ &= \frac{1}{s}\frac{Y(s)}{Z(s)} \end{aligned} \tag{107}$$

where

$$Y(s) = I_0(\lambda r) \quad ; \quad Z(s) = I_0(\lambda a) \tag{108}$$

Using the (temporary) substitution $\lambda = i\beta$, (98) may be written

$$\overline{F} = \frac{J_0(\beta r)}{J_0(\beta a)} \tag{109}$$

However $J_0(\zeta)$ is an entire function having only real zeros. Thus the only singularities of $\overline{F}$ are poles at the real roots, βn of

$$J_0(\beta a) = 0 \tag{110}$$

The corresponding values of s are:

$$S_n = -\alpha\left(\mu^2 + \beta_n^2\right) \tag{111}$$

Following Lowan, we write the desired inverse as

$$\begin{aligned} G_\mu(a, r; t) &= \frac{Y(0)}{Z(0)} + \sum_{n=1}^{\infty} \frac{Y(s_n)\, e^{s_n t}}{S_n \left[\frac{\partial Z}{\partial S}\right]_{S=S_n}} \\ &= \frac{I_0(\mu r)}{I_0(\mu a)} - \frac{2}{a} \sum_{n=1}^{\infty} \frac{\beta_n J_0(\beta_n r)\, e^{-\alpha(\mu^2+\beta_n^2)t}}{(\beta_n^2+\mu^2)\, J_1(\beta_n a)} \end{aligned} \tag{112}$$

Thus taking the derivative of (112) and substituting into (106) yields.

$$\begin{aligned} W(r,x,t) = \frac{4\alpha}{\pi a} \sum_{n=1}^{\infty} \beta_n \frac{J_0(\beta_n r)}{J_1(\beta_n a)} \int_0^{\infty}\int_0^{t} e^{-\alpha\beta_n^2 \eta} \cdot g(\xi, t-\eta) \cdot \\ \cdot \left[\int_0^{\infty} e^{-\alpha\mu^2\eta} \cdot \sin\mu x \sin\mu\xi \, d\mu \right] d\eta\, d\xi \end{aligned} \tag{113}$$

It is possible to carry out the integration with respect to μ in (113) and obtain

$$
u(r,x,t) = \sum_{n=1}^{\infty} R_2(r) \int_0^{\infty}\int_0^{t} \phi(\xi, t-\eta)\, K_1(x,\xi,t,\eta)\, d\eta\, d\xi \tag{114}
$$

where

$$
R_2(r) = \frac{J_0(\beta_n r)}{\beta_n J_1(\beta_n a)} \tag{115}
$$

$$
K_1(x,\xi,t,\eta) = \sqrt{\frac{\alpha}{\pi}} \cdot \frac{\beta_n^2}{\alpha} \frac{1}{\sqrt{\eta}}\, e^{\frac{V}{2\alpha}(x-\xi)} \cdot e^{-\eta\left(\frac{V^2}{4\alpha}+\alpha\beta_n^2\right)} \cdot \left\{ \exp\left[\frac{-(\xi-x)^2}{4\alpha\eta}\right] - \exp\left[\frac{-(\xi+x)^2}{4\alpha\eta}\right]\right\} \tag{116}
$$

Note that $K_1(x,\xi,t,\eta)$ given in (116) differs from $K_1(x,\xi,t,\eta)$ given by (50) only in the eigenvalues; otherwise they are identical.

For later purposes we need the partial derivative of (114) with respect to r, evaluated at r = a.

$$
\left[\frac{\partial u}{\partial r}\right]_{r=a} = -\sum_{n=1}^{\infty} \int_0^{\infty}\int_0^{t} \phi(\xi, t-\eta)\, K_1(x,\xi,t,\eta)\, d\xi\, d\eta \tag{117}
$$

3. Solid Pipe; Homogeneous Boundary Condition at Interface

Consider the problem stated mathematically as

$$
\frac{\partial^2 u}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial u}{\partial t} \tag{118}
$$

with the boundary and initial conditions:

$$u = 1 \qquad x = 0 \quad , \quad a \leqq r \leqq b \quad , \quad t \geqq 0 \tag{119}$$

$$u = 0 \qquad x \to \infty \quad , \quad a \leqq r \leqq b \quad , \quad t \geqq 0 \tag{120}$$

$$\frac{\partial u}{\partial r} = 0 \qquad 0 \leqq x < \infty \quad , \quad r = b \quad , \quad t \geqq 0 \tag{121}$$

$$u = 0 \qquad 0 \leqq x < \infty \quad , \quad r = a \quad , \quad t \geqq 0 \tag{122}$$

$$u = 0 \qquad 0 \leqq x < \infty \quad , \quad a \leqq r \leqq b \quad , \quad t = 0 \tag{123}$$

We first define the following notational device for a linear combination of Bessel functions of the first and second kind of an integer order.

Let

$$_{k}\mathcal{C}_{n}(r, b; \gamma) = J_{k}(\gamma r) Y_{n}(\gamma b) - Y_{k}(\gamma r) J_{n}(\gamma b) \tag{124}$$

Nowassume a solution of the form

$$u = A \, {}_{0}\mathcal{C}_{1}(r, b; \gamma) \, X_{2}(x, t) \tag{125}$$

Substitution of (125) in (118) yields

$$\frac{\partial^2 X_2}{\partial x^2} - \gamma^2 X_2 = \frac{1}{\alpha} \frac{\partial X_2}{\partial t} \tag{126}$$

The function (125) already satisfies (121), i.e.,

$$\left[\frac{\partial\, {}_0\mathcal{E}_1(r,b;\gamma)}{\partial r}\right]_{r=b} = \gamma\, {}_1\mathcal{E}_1(b,b;\gamma) = 0 \tag{127}$$

Condition (122) yields the characteristic equation

$${}_0\mathcal{E}_1(a,b;\gamma) = 0 \tag{128}$$

the roots of which are denoted by γ_n, n = 1, 2, ... Because of linearity we can therefore write

$$u = \sum_{n=1}^{\infty} A_n\, {}_0\mathcal{E}_1(r,b;\gamma)\, X_2(x,t) \tag{129}$$

To solve (126) we take Laplace transform with respect to time. This yields a second order ordinary homogeneous differential equation, the solution of which is

$$\bar{X}_2(x,s) = B \exp\left[-x\sqrt{\gamma_n^2 + s/\alpha}\right] \tag{130}$$

In (130) the exponential with positive exponent has been discarded so that condition (120) may be satisfied. Substitution of (130) in (129) and absorbing B into A_n, we obtain

$$\bar{u}(r,x,s) = \sum_{n=1}^{\infty} A_n\, {}_0\mathcal{E}_1(r,b;\gamma) \exp\left[-x\sqrt{\gamma_n^2 + s/\alpha}\right] \tag{131}$$

Thus at this point we observe that the inverse of (131) satisfies (118), (120), (121), (122) and (123). Condition (119) provides a means of evaluating A_n. Taking x = 0 (119) gives

$$\frac{1}{s} = \sum_{n=1}^{\infty} A_n \, {}_0\mathcal{C}_1 \, (r, b; \gamma_n) \tag{132}$$

It is not too difficult to show that (124) has the usual orthogonality properties of Bessel functions with the weighting function r. Thus we merely present the result as

$$\int_a^b r \, {}_0\mathcal{C}_1(r,b;\gamma_n) \, {}_0\mathcal{C}_1(r,b;\gamma_m) \, dr = \frac{\frac{2a}{\gamma_m} \, {}_1\mathcal{C}_1 \, (a,b;\gamma_m)}{a^2 \, {}_1\mathcal{C}_1^2(a,b;\gamma_m) - b^2 \, {}_0\mathcal{C}_1^2(b,b;\gamma_m)} \tag{133}$$

if n = m and zero if n ≠ m. Use of this orthogonality property yields

$$A_n = \frac{1}{s}\left[\frac{\frac{2a}{\gamma_n} \, {}_1\mathcal{C}_1 \, (a,b;\gamma_n)}{a^2 \, {}_1\mathcal{C}_1^2(a,b;\gamma_n) - b^2 \, {}_0\mathcal{C}_1^2(b,b;\gamma_n)}\right] \tag{134}$$

Substitution of (134) in (131) yields

$$\bar{u} = \sum_{n=1}^{\infty}\left[\frac{\frac{2a}{\gamma_n} \, {}_1\mathcal{C}_1(a,b;\gamma_n) \, {}_0\mathcal{C}_1 \, (r,b;\gamma_n)}{a^2 \, {}_1\mathcal{C}_1^2(a,b;\gamma_n) - b^2 \, {}_0\mathcal{C}_1^2(b,b;\gamma_n)}\right]\left[\frac{1}{s} \exp\left(-x\sqrt{\gamma_n^2 + \frac{s}{\alpha}}\right)\right] \tag{135}$$

Note that the parameter s appears only in the last term. The inverse of (135) is tabulated. Hence,

$$u \, (r, x, t) = \sum_{n=1}^{\infty} R_3(r) \, X_2(x, t) \tag{136}$$

where

$$R_3(r) = \frac{a}{\gamma_n} \, \frac{{}_1\mathcal{C}_1(a,b;\gamma_n) \, {}_0\mathcal{C}_1 \, (r,b;\gamma_n)}{a^2 \, {}_1\mathcal{C}_1^2(a,b;\gamma_n) - b^2 \, {}_0\mathcal{C}_1^2(b,b;\gamma_n)} \tag{137}$$

$$X_2(x,t) = e^{-\gamma_n x} \operatorname{erfc}\left(\frac{x - 2\alpha\gamma_n t}{2\sqrt{\alpha t}}\right) + e^{\gamma_n x} \operatorname{erfc}\left(\frac{x + 2\alpha\gamma_n t}{2\sqrt{\alpha t}}\right) \tag{138}$$

The partial derivative of (136) with respect to r and evaluated at r = a is

$$\left[\frac{\partial u}{\partial r}\right]_{r=a} = \sum_{n=1}^{\infty} R_3'(a) \cdot X_2(x,t) \tag{139}$$

where

$$R_3'(a) = \frac{1/a}{\left[\frac{b}{a}\,\frac{{}_0\mathcal{C}_1(b,b;\gamma_n)}{{}_0\mathcal{C}_1(a,b;\gamma_n)}\right]^2 - 1} \tag{140}$$

This result will also be used later.

4. <u>Solid Wall; Non-homogeneous Boundary Condition at Interface</u>

We consider the immediately preceding problem with the following changes in boundary conditions

$u = 0$ (rather than u = 1) $x = 0$ $a \leqq r \leqq b$, $t \geqq 0$

$u = \psi(x,t)$ (rather than u = 0) $x = 0$ $r = a$, $t \geqq 0$

where $|\psi(x,t)| \leqq M < \infty$ for all x and t. After Laplace transformation we have

$$\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{u}}{\partial r}\right) = \frac{s}{\alpha}\bar{u} \tag{141}$$

with the conditions

$$\bar{u} = 0 \qquad x = 0 \qquad a \leqq r \leqq b \tag{142}$$

$$\bar{u} < \infty \qquad x \to \infty \qquad a \leqq r \leqq b \tag{143}$$

$$\frac{\partial \bar{u}}{\partial r} = 0 \qquad 0 \leqq x < \infty \quad , \quad r = b \tag{144}$$

$$\bar{u} = \bar{\Psi}(x, s) \qquad 0 \leqq x < \infty \quad , \quad r = a \tag{145}$$

The solution of (141) may be constructed following the procedure used by Lowan. Thus, consider the function

$$\bar{u} = \frac{2}{\pi} \int_0^\infty \int_0^\infty \bar{\bar{F}}(a, b, r; \lambda) \bar{\Psi}(\xi, s) \sin \mu x \sin \mu \xi \, d\mu \, d\xi \tag{146}$$

where

$$\bar{F}(a, b, r; \lambda) = \frac{I_0(\lambda r) K_1(\lambda b) + K_0(\lambda r) I_1(\lambda b)}{I_0(\lambda a) K_1(\lambda b) + K_0(\lambda a) I_1(\lambda b)} \tag{147}$$

and

$$\lambda^2 = \mu^2 + s/\alpha \tag{148}$$

(I_n and K_n are modified Bessel functions of first and second kind of order n). Observe that μ and ξ are dummy variables of integration.

Direct substitution shows that (141) is satisfied. It is also easy to verify that all the boundary conditions are satisfied. In particular we note that

$$\bar{u}(a, x, s) = \frac{2}{\pi} \int_0^\infty \int_0^\infty \bar{\Psi}(\xi, s) \sin \mu x \sin \mu \xi \, d\mu \, d\xi \tag{149}$$

$$= \bar{\Psi}(x, s)$$

by the Fourier identity. As regards boundedness,

$$\left|\bar{u}(r,x,s)\right| \leq \frac{2M}{\pi}\int_0^\infty\int_0^\infty \sin\mu x \sin\mu\xi \, d\mu \, d\xi = \frac{2M}{\pi} < \infty \tag{150}$$

This completes the proof that (146) is the desired solution of (141) and satisfies all the auxiliary conditions. Again we follow the procedure indicated by Lowan, and the sequence of analysis is precisely that employed in subsection IIIB2 for the case of slab geometry; the present case is different only in details. Accordingly, we abbreviate the treatment and write immediately,

$$u(r,x,t) = \frac{2}{\pi}\int_0^\infty\int_0^\infty \sin\mu x \sin\mu\xi \, d\mu \, d\xi \int_0^t \frac{\partial G_\mu(a,b,r;\eta)}{\partial\eta}\psi(\xi,t-\eta)\,d\eta \tag{151}$$

and

$$G_\mu(a,b,r;t) = \mathcal{L}^{-1}\left[\frac{1}{s}\bar{F}(a,b,r;\lambda)\right] = \mathcal{L}^{-1}\left[\frac{1}{s}\,\frac{Y(s)}{Z(s)}\right] \tag{152}$$

where

$$Y(s) = I_0(\lambda r)\,K_1(\lambda b) + K_0(\lambda r)\,I_1(\lambda b) \tag{153}$$

$$Z(s) = I_0(\lambda a)\,K_1(\lambda b) + K_0(\lambda a)\,I_1(\lambda b) \tag{154}$$

Using the temporary substitution $\lambda = i\gamma$, (147) may be written

$$\bar{F}(a,b,r;\gamma) = \frac{J_0(\gamma r)Y_1(\gamma b) - J_1(\gamma b)Y_0(\gamma r)}{J_0(\gamma a)Y_1(\gamma b) - J_1(\gamma b)Y_0(\gamma a)} = \frac{{}_0\mathcal{E}_1(r,b;\gamma)}{{}_0\mathcal{E}_1(a,b;\gamma)} \tag{155}$$

where we have used the definition (124). It is reasonable to assume that the function ${}_0\mathcal{C}_1$ (a, b; γ) considered as a function of γ has only real zeros. Thus the only singularities of $\overline{F}$ are poles at the real roots, γ_n, of

$$ {}_0\mathcal{C}_1(a, b; \gamma_n) = 0 \tag{156} $$

The corresponding values of s are.

$$ s_n = -\alpha\left(\gamma_n^2 + \mu^2\right) \tag{157} $$

Following the procedure used by Lowan, for the inversion of (152), we obtain

$$ G_\mu(a, b, r; t) = \frac{Y(0)}{Z(0)} + \sum_{n=1}^{\infty} \frac{Y(s_n)\, e^{s_n t}}{s_n \left[\dfrac{\partial Z}{\partial s}\right]_{s=s_n}} \tag{158} $$

$$ G_\mu(a, b, r; t) = \frac{I_0(\mu r) K_1(\mu b) + K_0(\mu r) I_1(\mu b)}{I_0(\mu a) K_1(\mu b) + K_0(\mu a) I_1(\mu a)} + 2\sum_{n=1}^{\infty} \frac{\gamma_n}{\gamma_n^2+\mu^2} \; \frac{{}_0\mathcal{C}_0(r, b; \gamma_n)\, e^{-\alpha(\gamma_n^2+\mu^2)t}}{\left[b\, {}_0\mathcal{C}_0(a, b; \gamma_n) + a\, {}_1\mathcal{C}_1(a, b; \gamma_n)\right]} \tag{159} $$

Thus taking the derivative of (159) and substituting into (151) yields.

$$ u(r, x, t) = \sum_{n=1}^{\infty} R_4(r) \int_0^{\infty}\!\!\int_0^{t} \Psi(\xi, t-\eta)\, K_2(x, \xi, t, \eta)\, d\eta\, d\xi \tag{160} $$

where

$$R_4(r) = -\frac{\gamma_n \, {}_0\mathscr{E}_1(r, b; \gamma_n)}{b \, {}_0\mathscr{E}_0(a, b; \gamma_n) + a \, {}_1\mathscr{E}_1(a, b; \gamma_n)} \tag{161}$$

$$K_2(x, \xi, t, \eta) = \sqrt{\frac{\alpha}{\pi}} \cdot \frac{e^{-\alpha \gamma_n^2 \eta}}{\sqrt{\eta}} \left\{ \exp\left[\frac{-(\xi - x)^2}{4\alpha\eta}\right] - \exp\left[\frac{-(\xi + x)^2}{4\alpha\eta}\right] \right\} \tag{162}$$

For future use we require the following

$$\left[\frac{\partial u}{\partial r}\right]_{r=a} = \sum_{n=1}^{\infty} R_4'(a) \int_0^{\infty}\int_0^{t} \psi(\xi, t-\eta) \, K_2(x, \xi, t, \eta) \, d\eta \, d\xi \tag{163}$$

where

$$R_4'(\alpha) = \frac{\gamma_n^2}{b \dfrac{{}_0\mathscr{E}_0(a, b; \gamma_n)}{{}_1\mathscr{E}_1(a, b; \gamma_n)} + a} \tag{164}$$

5. Combination of Solutions (Cylindrical Geometry)

a. For the Fluid

Simple superposition of the solutions (88) and (114) yields

$$\theta(r, x, t) = \sum_{n=1}^{\infty} \left[R_1(r) X_1(x, t) + R_2(r) \int_0^{\infty}\int_0^{t} \phi(\xi, t-\eta) K_1(x, \xi, t, \eta) \, d\eta \, d\xi \right] \tag{165}$$

where having restored the appropriate subscripts

$$R_1(r) = R_2(r) = \frac{1}{\beta_n} \frac{J_0(\beta_n r)}{J_1(\beta_n r)} \tag{166}$$

$$X_1(x,t) = \frac{1}{\alpha} e^{\frac{Vx}{2\alpha_f}} \left[e^{-\omega_n x} \cdot \operatorname{erfc}\left(\frac{x - 2\omega_n \alpha_f t}{2\sqrt{\alpha_f t}}\right) + e^{\omega_n x} \cdot \operatorname{erfc}\left(\frac{x + 2\omega_n \alpha_f t}{2\sqrt{\alpha_f t}}\right)\right] \tag{167}$$

$$K_1(x,\xi,t,\eta) = \sqrt{\frac{\alpha_f}{\pi}} \frac{\beta_n^2}{\sqrt{\eta}} e^{\frac{V}{2\alpha_f}(x-\xi)} \cdot e^{-\eta(\alpha_f \beta_n^2 + V^2/4\alpha_f)} \cdot \left\{ \exp\left[\frac{-(\xi - x)^2}{4\alpha_f}\right] - \exp\left[\frac{-(\xi + x)^2}{4\alpha_f \eta}\right]\right\} \tag{168}$$

Similarly superposing (91) and (117) we obtain

$$\left[\frac{\partial \theta}{\partial r}\right]_{r=a} = -\sum_{n=1}^{\infty} \left[X_1(x,t) + \int_0^{\infty}\int_0^{t} \Phi(\xi, t-\eta)\, K_1(x,\xi,t,\eta)\, d\eta\, d\xi \right] \tag{169}$$

b. For the Solid Pipe

Again simple superposition of (136) and (160) yields

$$T(r,x,t) = \sum_{n=1}^{\infty} \left[R_3(r) X_2(x,t) + R_4(r) \int_0^{\infty}\int_0^{t} \Psi(\xi, t-\eta) K_2(x,\xi,t,\eta)\, d\eta\, d\xi \right] \tag{170}$$

where again restoring the appropriate subscripts

$$R_3(r) = \frac{a}{\gamma_n} \frac{{}_1\mathcal{E}_1(a,b;\gamma_n)\, {}_0\mathcal{E}_1(r,b;\gamma_n)}{a^2\, {}_1\mathcal{E}_1^2(a,b;\gamma_n) - b^2\, {}_0\mathcal{E}_1^2(b,b;\gamma_n)} \tag{171}$$

$$R_4(r) = -\frac{b}{\gamma_n} \frac{{}_0\mathcal{E}_1(r,b;\gamma_n)}{b\, {}_0\mathcal{E}_0(a,b;\gamma_n) + a\, {}_1\mathcal{E}_1(a,b;\gamma_n)} \tag{172}$$

$$K_2(x,\xi,t,\eta) = \sqrt{\frac{\alpha_m}{\pi}} \frac{\gamma_n^2}{b} \cdot \frac{e^{-\alpha_m \gamma_n^2 \eta}}{\sqrt{\eta}} \cdot \left\{ \exp\left[\frac{-(\xi - x)^2}{4\alpha_m \eta}\right] - \exp\left[\frac{-(\xi + x)^2}{4\alpha_m \eta}\right]\right\} \tag{173}$$

similarly superposing (139) and (163) yields

$$\left[\frac{\partial T}{\partial r}\right]_{r=a} = \sum_{n=1}^{\infty}\left[R_3'(a)\,X_2(x,t) + R_4'(a)\int_0^{\infty}\int_0^{t}\Psi(\xi,t-\eta)K_2(x,\xi,t,\eta)\,d\eta\,d\xi\right] \tag{174}$$

where

$$R_3'(a) = \frac{1}{\left[\frac{b}{a}\,\frac{{}_0\mathcal{E}_1(b,b;\gamma_n)}{{}_1\mathcal{E}_1(a,b;\gamma_n)}\right]^2 - 1} \tag{175}$$

$$R_4'(a) = \frac{\gamma_n^2}{\frac{b\,{}_0\mathcal{E}_0(a,b;\gamma_n)}{{}_1\mathcal{E}_1(a,b;\gamma_n)} + a} \tag{176}$$

E. DERIVATION OF INTEGRAL EQUATION (CYLINDRICAL GEOMETRY)

We now proceed as we did in section IIIC for the case of slab geometry. Equations (165) and (170) give the fluid and solid temperature distribution each in terms of the arbitrarily preassigned function $\phi(x,t)$ and $\Psi(x,t)$. If we now return to the actual boundary condition, namely that of continuity of heat flux at the interface, we have

$$-K_f\left[\frac{\partial \theta}{\partial r}\right]_{r=a} = h\,(\phi - \Psi) \tag{177}$$

and

$$-K_m\left[\frac{\partial T}{\partial r}\right]_{r=a} = h\,(\phi - \Psi) \tag{178}$$

and solving (177) for ψ in terms of ϕ , we obtain.

$$\psi = \frac{K_f}{h}\left[\frac{\partial\theta}{\partial r}\right]_{r=a} + \phi \tag{179}$$

Substitution of (169) into (179) yields.

$$\psi = \phi - \frac{K_f}{h}\sum_{n=1}^{\infty}\left[X_1(x,t) + \int_0^{\infty}\int_0^{t}\phi(\xi,t-\eta)K_1(x,\xi,t,\eta)\,d\eta\,d\xi\right] \tag{180}$$

Further substitution for ψ from (180) above into (174) and the result into (178) yields

$$\begin{aligned} K_m\sum_{n=1}^{\infty}\Bigg\{ & R_3'(a)\,X_2(x,t) + R_4'(a)\int_0^{\infty}\int_0^{t}K_2(x,\xi,t,\eta)\cdot \\ & \cdot\Bigg\{\phi(\xi,t-\eta) - \frac{K_f}{h}\sum_{m=1}^{\infty}\Bigg[X_1(\xi,t-\eta) + \int_0^{\infty}\int_0^{t-\eta}\phi(\beta,t-\eta-\alpha)\cdot \\ & \cdot K_1(\xi,\beta,t-\eta,\alpha)\,d\alpha\,d\beta\Bigg]\Bigg\}\Bigg\}d\eta\,d\xi + K_f\sum_{n=1}^{\infty}\left[X_1(x,t) + \int_0^{\infty}\int_0^{t}\phi(\xi,t-\eta)K_1(x,\xi,t,\eta)\,d\eta\,d\xi\right] = 0 \end{aligned} \tag{181}$$

The same remarks as those following equation (70) for the integral equation of slab geometry may be made here too. Similarly therefore if we let h approach infinity, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty}\Bigg\{ & \frac{K_m}{K_f}R_3'(a)\,X_2(x,t) + X_1(x,t) + \int_0^{\infty}\int_0^{t}\phi(\xi,t-\eta)\cdot \\ & \left[K_1(x,\xi,t,\eta) + \frac{K_m}{K_f}R_4'(a)\,K_2(x,\xi,t,\eta)\right]d\eta\,d\xi\Bigg\} = 0 \end{aligned} \tag{182}$$

This is equivalent to (71). Note that (182) can be written more compactly as

$$\sum_{n=1}^{\infty}\left[X^{*}(x,t)+\int_{0}^{\infty}\int_{0}^{t}\phi(\xi,t-\eta)\,K^{*}(x,\xi,t,\eta)\,d\eta\,d\xi\right]=0 \tag{183}$$

where

$$X^{*}(x,t)=\frac{K_m}{K_f}R_3'(a)\,X_2(x,t)+X_1(x,t) \tag{184}$$

$$K^{*}(x,\xi,t,\eta)=K_1(x,\xi,t,\eta)+\frac{K_m}{K_f}R_4'(a)\,K_2(x,\xi,t,\eta) \tag{185}$$

A similar abbreviation may be made for (71).

The reader should note that (70) and (181) are of exactly the same form; their only difference is in the particular form assumed by the kernel and driving function and in the fact that the counterparts of $R_3'(a)$ and $R_4'(a)$ appearing here are simply unity in the case of (70).

F. REMARKS ON SOLVING THE INTEGRAL EQUATION

It is obvious that the integral equation (70) or (71) for the case of slab geometry, or (181) or (182) for the case of cylindrical geometry will not be solved easily. It involves multiple integrals, one with variable upper limit and one with fixed infinite upper limit. The kernel and driving functions, although well behaved, are quite complicated.

One of the ways of dealing with such a complicated problem is to do what has been done in this thesis, namely, make additional assumptions so as to simplify the problem to the point that it becomes analytically tractable. Our attention has been exclusively directed in this way and we have made no serious attempt to deal with these fundamental integral equations. However, the following remarks and conjectures may be of interest.

All the analysis herein shows that while $\phi(x, t)$ is a smooth function, almost all its variation takes place in the immediate neighborhood of the wave front $x = Vt$. It is conjectured that a contour map of $\phi(x, t)$ would be similar to that shown in Figure 2. In this figure the contours are shown as straight lines merely because the

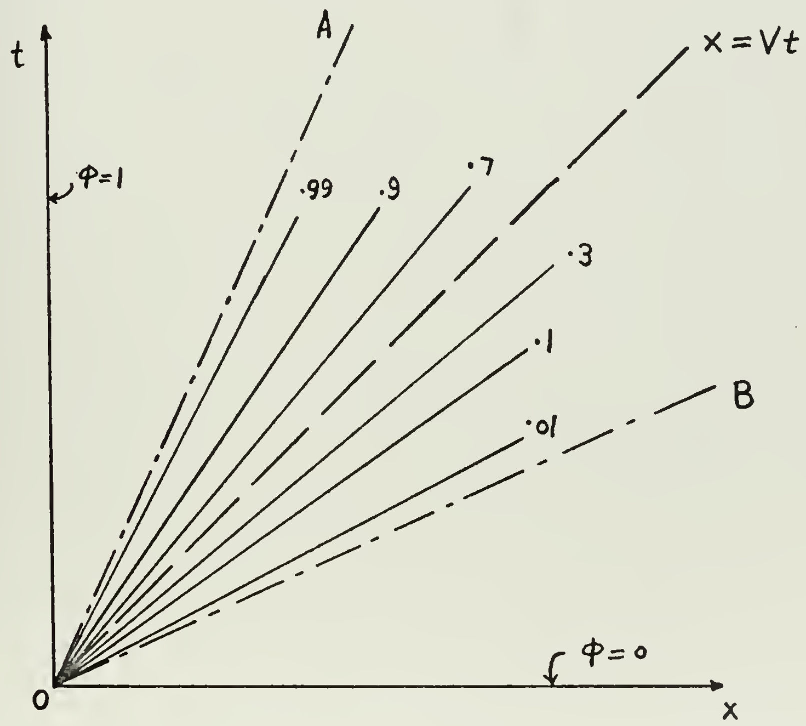


Figure 2. Conjectured Contours of $\phi(x, t)$.

nature and degree of curvature are unknown. It is tempting to consider replacing the boundary and initial conditions $\phi = 1$ along $x = 0$ and $\phi = 0$ along $t = 0$, by the conditions $\phi = 1$ along OA and $\phi = 0$ along OB so as to provide a field in which the "action" of $\phi(x, t)$ is more evenly distributed. This might perhaps be useful in an attempt at solution via Gallerkin's (or a similar) method in which a solution is assumed which involves unknown scalar constants and these are determined so as to minimize an error functional. In such a procedure, it is evident that simplifications would result from assuming the contour lines to be straight.

IV. METHOD OF PERTURBATION

In this section we employ the method of perturbation to treat the problem described by equations (II-10) and (II-11) and the associated boundary and initial conditions. The reader will recall that these equations resulted from having taken radial (or transverse) averages of temperatures in the fluid and in the pipe wall. Thus radial (or transverse) conductivity is, in effect, absorbed into the surface heat transfer coefficient h^* defined in equation (II-9). The influence of axial conductivity appears explicitly in the following analysis.

An outline of what will be accomplished in this section may be helpful to the reader. First, an ordinary perturbation expansion is made in terms of the parameter $\epsilon = 1/V$. It turns out that only even powers of ϵ are involved. The zeroth order solutions are identical to well known solutions described in section 15-3 of [7]. The effect of longitudinal conductivity does not appear in this zero order solution but it does appear in subsequent solutions. It is clear from the sequence of higher order perturbations that the solutions fail at and ahead of the wave front, $x = Vt$. Severe singularities are encountered at the wave front and the solutions are identically zero ahead of the wave front.

Thus a singular perturbation development is introduced to explain the behavior ahead of the front. The lowest order solution for this

singular perturbation expansion is similar to the known solution for a moving fluid, with diffusion and with an isothermal boundary condition. Terms in this singular perturbation development involve unknown functions of time which are determined by a process of matching with the solutions valid behind the front.

It should be remarked that the system of equations dealt with in this section is obtainable from either slab geometry or cylindrical geometry. The process of radial or transverse averaging of temperatures erases the distinction between the two geometries.

A. ORDINARY PERTURBATION ANALYSIS

Equation (II-10) and (II-11) together with associated conditions are repeated here for reference purposes. The notation has been slightly changed; we omit the asterisks which indicated radial averages and we introduce the convenient constants b_1 and b_2 defined below.

In the fluid, we have

$$\alpha_f \frac{\partial^2 \theta}{\partial x^2} - V \frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial t} + b_1 (\theta - T) \tag{1}$$

where

$$b_1 = \frac{h P}{\rho_f c_f A_f} \tag{2}$$

subject to the conditions

$$\begin{cases} \theta = 1 & x = 0 \;, \quad t \geqq 0 \\ \theta \to 0 & x \to \infty \;, \quad t \geqq 0 \\ \theta = 0 & 0 \leqq x < \infty \;, \quad t = 0 \end{cases} \tag{3}$$

For the solid we have

$$\alpha_m \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} - b_2(\theta - T) \tag{4}$$

where

$$b_2 = \frac{h P}{\rho_m c_m A_m} \tag{5}$$

subject to the conditions.

$$\begin{cases} T = 1 & x = 0 \;, \quad t \geqq 0 \\ T \to 0 & x \to \infty \;, \quad t \geqq 0 \\ T = 0 & 0 \leqq x < \infty \;, \quad t = 0 \end{cases} \tag{6}$$

It is convenient to normalize with the following change of variables

$$\begin{cases} t = t \\ \eta = x/\nu = \epsilon x \quad ; \quad \epsilon = 1/\nu \end{cases} \tag{7}$$

so that equation (1) and conditions (3) become

$$\alpha_f \epsilon^2 \frac{\partial^2 \theta}{\partial \eta^2} - \frac{\partial \theta}{\partial \eta} = \frac{\partial \theta}{\partial t} + b_1(\theta - T) \tag{8}$$

$$\begin{cases} \theta = 1 & \eta = 0 \quad , & t \geqq 0 \\ \theta \to 0 & \eta \to \infty \quad , & t \geqq 0 \\ \theta = 0 & 0 \leqq \eta \leqq \infty & t = 0 \end{cases} \tag{9}$$

and similarly equation (4) and conditions (6) become

$$\alpha_m \epsilon^2 \frac{\partial^2 T}{\partial \eta^2} = \frac{\partial T}{\partial t} - b_2(\theta - T) \tag{10}$$

$$\begin{cases} T = 1 & \eta = 0 & t \geqq 0 \\ T \to 0 & \eta \to \infty & t \geqq 0 \\ T = 0 & 0 \leqq \eta < \infty & t = 0 \end{cases} \tag{11}$$

Equations (8) and (10) with conditions (9) and (11) can be approached by the use of Laplace Transformations as was done by Hiep [13], but there seems to be no way of solving the transformed equations or performing the inversion other than by strictly numerical procedures.

This system of equations may be considered as depending on the parameter ϵ^2 which appears in (8) and (10). For large values of V, this parameter is small so that it is reasonable to consider a perturbation expansion in ϵ^2.

(Note that ϵ itself does not appear in (8) through (11), only ϵ^2; thus the expansion we develop is in powers of ϵ^2, that is, only even powers of ϵ are involved. There are other ways of showing that odd powers should not appear in this perturbation expansion, but the one given here is sufficient.) Thus,

$$\theta(\eta, t; \epsilon^2) = \theta_0(\eta, t) + \epsilon^2 \theta_1(\eta, t) + \cdots \tag{12}$$

$$T(\eta, t; \epsilon^2) = T_0(\eta, t) + \epsilon^2 T_1(\eta, t) + \cdots \tag{13}$$

Substitution of (12) and (13) in (8) and (10) and equating powers of ϵ^2, the following sequence of equations are obtained

$$\frac{\partial \theta_0}{\partial \eta} + \frac{\partial \theta_0}{\partial t} + b_1(\theta_0 - T_0) = 0 \tag{14}$$

$$\frac{\partial \theta_1}{\partial \eta} + \frac{\partial \theta_1}{\partial t} + b_1(\theta_1 - T_1) = \alpha_f \frac{\partial^2 \theta_0}{\partial \eta^2} \tag{15}$$

$$\frac{\partial \theta_i}{\partial \eta} + \frac{\partial \theta_i}{\partial t} + b_1(\theta_i - T_i) = \alpha_f \frac{\partial^2 \theta_{i-1}}{\partial \eta^2}, \quad i \geqq 2 \tag{16}$$

for the fluid and

$$\frac{\partial T_0}{\partial t} - b_2(\theta_0 - T_0) = 0 \tag{17}$$

$$\frac{\partial T_1}{\partial t} - b_2(\theta_1 - T_1) = \alpha_m \frac{\partial^2 T_0}{\partial \eta^2} \tag{18}$$

$$\frac{\partial T_i}{\partial t} - b_2(\theta_i - T_i) = \alpha_m \frac{\partial^2 T_{i-1}}{\partial \eta^2} \quad , \quad i \geqq 2 \tag{19}$$

for the solid. The boundary and initial conditions are

$$\begin{cases} \theta_0(\eta,t) = 1 & \eta = 0 \;, \quad t \geqq 0 \\ \theta_0(\eta,t) = 0 & 0 \leqq \eta < \infty \;, \quad t = 0 \end{cases} \tag{20}$$

$$\left.\begin{cases} \theta_i(\eta,t) = 0 & \eta = 0 \;, \quad t \geqq 0 \\ \theta_i(\eta,t) = 0 & 0 \leqq \eta < \infty \quad t = 0 \end{cases}\right\}, \; i \geqq 1 \tag{21}$$

and

$$\begin{cases} T_0(\eta,t) = 1 & \eta = 0 \;, \quad t \geqq 0 \\ T_0(\eta,t) = 0 & 0 \leqq \eta < \infty \;, \quad t = 0 \end{cases} \tag{22}$$

$$\left.\begin{cases} T_i(\eta,t) = 0 & \eta = 0 \;, \quad t \geqq 0 \\ T_i(\eta,t) = 0 & 0 \leqq \eta < \infty \;, \quad t = 0 \end{cases}\right\} \; i \geqq 1 \tag{23}$$

Since the order of these equations is less than that of the original equations, only one boundary condition can be imposed. It is not expected that the solutions will be uniformly valid. Starting with (14) and (17), we take the Laplace transformation with respect to t. The transformed equations are

$$\frac{\partial \bar{\theta}_0}{\partial \eta} + s\,\bar{\theta}_0 + b_1\left(\bar{\theta}_0 - \bar{T}_0\right) = 0 \tag{24}$$

$$(s + b_2)\,\bar{T}_0 = b_2\,\bar{\theta}_0 \tag{25}$$

Expressing $\bar{T}_0$ in (25) in terms of $\bar{\theta}_0$, substituting in (24), and solving the first order homogeneous differential equation yields

$$\bar{T}_0(\eta, s) = \frac{b_2}{s(s+b_2)} \exp\left[-\eta\left(s + b_1 - \frac{b_1 b_2}{s + b_2}\right)\right] \tag{26}$$

$$\bar{\theta}_0(\eta, s) = \frac{1}{s} \exp\left[-\eta\left(s + b_1 - \frac{b_1 b_2}{s + b_2}\right)\right] \tag{27}$$

The procedure for inversion of (26) and (27) is as indicated in Example 7-5 of [2]. The results are

$$\theta_0(\eta, t) = 0 \qquad \text{for} \quad t < \eta \tag{28}$$

$$\theta_0(\eta, t) = e^{-b_1\eta}\left[e^{-b_2\tau} \cdot I_0(2\sqrt{a\tau}) + b_2\int_0^{\tau} e^{-b_2 y} I_0(2\sqrt{ay})\,dy\right] \qquad \text{for} \quad t \geq \eta \tag{29}$$

and

$$T_0(\eta, t) = 0 \qquad \text{for} \quad t < \eta \tag{30}$$

$$T_0(\eta, t) = b_2 e^{-b_1\eta} \int_0^{\tau} e^{-b_2 y} I_0(2\sqrt{ay})\,dy \qquad \text{for} \quad t \geq \eta \tag{31}$$

where $a = b_1 b_2 \eta$ and $\tau = t - \eta$. Note the use of a in this sense; it should not be confused with a linear dimension. It may easily be shown that $\theta_0(\eta, t)$ and $To(\eta, t)$ both tend to unity as $t \to \infty$.

It is clear that the two solutions obtained for $T_0(\eta, t)$ and $\theta_0(\eta, t)$ have a wave like character with no disturbance for $\eta > t$. It is well to pause a moment to refer back to the physics of the problem. For fixed η (i. e., fixed value of x) no disturbance takes place until $t \geqq \eta$, or for fixed t, there is no disturbance for the region $\eta > t$. In what follows the condition $\tau = t - \eta = 0$ is referred to as the "wave front." The pipe wall temperature $T_0(\eta, t)$ is continuous at the wave front with discontinuous slope, while the fluid temperature $\theta_0(\eta, t)$ is itself discontinuous at the wave front.

The first corrections $\theta_1(\eta, t)$ and $T_1(\eta, t)$ are governed by equations similar to those for $\theta_0(\eta, t)$ and $T_0(\eta, t)$ except that these new **equations** are nonhomogeneous with the second derivative of $\theta_0(\eta, t)$ and $T_0(\eta, t)$ as forcing functions. These non-homogeneous terms vanish ahead of the wave front, and thus no correction ahead of the wave front is expected in the sequence of solutions θ_0, T_0, θ_1, T_1, ... In the next subsection we will demonstrate this by actually constructing the solutions θ_1 and T_1 and will see that the present perturbation expansion fails at the front.

B. HIGHER ORDER SOLUTIONS

The corrections $\theta_1(\eta, t)$ and $T_1(\eta, t)$ are also obtained by using Laplace transformation. From (18)

$$(s+b_2)\overline{T}_1 = b_2\overline{\theta}_1 + \alpha_m \frac{\partial^2 \overline{T}_0}{\partial \eta^2} \tag{32}$$

and from (26) we have

$$\frac{\partial^2 \overline{T}_0}{\partial \eta^2} = \frac{b_2}{s(s+b_2)}\left(s+b_1-\frac{b_1 b_2}{s+b_2}\right)^2 \cdot \exp\left[-\eta\left(s+b_1-\frac{b_1 b_2}{s+b_2}\right)\right] \tag{33}$$

Likewise from (15)

$$\frac{\partial \overline{\theta}_1}{\partial \eta} + (s+b_1)\overline{\theta}_1 = b_1\overline{T}_1 + \alpha_f \frac{\partial^2 \overline{\theta}_0}{\partial \eta^2} \tag{34}$$

and from (27)

$$\frac{\partial^2 \overline{\theta}_0}{\partial \eta^2} = \frac{1}{s}\left(s+b_1-\frac{b_1 b_2}{s+b_2}\right)^2 \cdot \exp\left[-\eta\left(s+b_1-\frac{b_1 b_2}{s+b_2}\right)\right] \tag{35}$$

Substituting (33) in (32) and solving for $\overline{T}_1$ in terms of $\overline{\theta}_1$ and then substituting the result in (34) results in a first order non-homogeneous differential equation for $\overline{\theta}_1(\eta, s)$.

$$\frac{\partial \overline{\theta}_1}{\partial \eta} + \left(s+b_1-\frac{b_1 b_2}{s+b_2}\right)\overline{\theta}_1 = s\left[\frac{\alpha_m b_1 b_2}{(s+b_2)^2} + \alpha_f\right]\left(1+\frac{b_1}{s+b_2}\right)^2 \exp\left[-\eta\left(s+b_1-\frac{b_1 b_2}{s+b_2}\right)\right] \tag{36}$$

The solution of (36) is easy to obtain.

$$\bar{\theta}_1 = s \cdot \eta \left[\frac{\alpha_m b_1 b_2}{(s+b_2)^2} + \alpha_f \right] \left(1 + \frac{b_1}{s+b_2} \right)^2 \exp\left[-\eta \left(s + b_1 - \frac{b_1 b_2}{s+b_2} \right) \right] \tag{37}$$

It is now merely a question of substitution of $\bar{\theta}_1$ from (37) and $\frac{\partial^2 \bar{T}_0}{\partial \eta^2}$ from (33) in (32) to obtain $\bar{T}_1(\eta, s)$. After rearranging we have

$$\bar{T}_1 = \frac{s b_2}{(s+b_2)} \left[\alpha_f \eta + \frac{\alpha_m b_1 b_2 \eta}{(s+b_2)^2} + \frac{\alpha_m}{s+b_2} \right] \left(1 + \frac{b_1}{s+b_2} \right)^2 \exp\left[-\eta \left(s + b_1 - \frac{b_1 b_2}{s+b_2} \right) \right] \tag{38}$$

To invert (37) it is convenient to express $\bar{\theta}_1(\eta, s)$ as follows

$$\bar{\theta}_1 = s\, e^{-\eta b_1} \cdot e^{-\eta s} \cdot e^{\frac{a}{s+b_2}} \sum_{n=0}^{4} k_n (s+b_2)^{-n} \tag{39}$$

where

$$\begin{aligned} k_o &= \alpha_f \eta \\ k_1 &= 2 \alpha_f b_1 \eta \\ k_2 &= \alpha_f \eta b_1^2 + \alpha_m b_1 b_2 \eta \\ k_3 &= 2 \alpha_m b_1^2 b_2 \eta \\ k_4 &= \alpha_m b_1^3 b_2 \eta \end{aligned} \tag{40}$$

Similarly

$$\bar{T}_1 = s\, e^{-\eta b_1} \cdot e^{-\eta s} \cdot e^{\frac{a}{s+b_2}} \sum_{n=1}^{5} k_n^* (s+b_2)^{-n} \tag{41}$$

where

$$
\begin{aligned}
k_1^* &= \alpha_f \eta b_2 \\
k_2^* &= 2\alpha_f b_1 b_2 \eta + \alpha_m b_2 \\
k_3^* &= b_1 b_2 \left(\alpha_f \eta b_1 + 2\alpha_m + \alpha_m b_2 \eta\right) \\
k_4^* &= \alpha_m b_1^2 b_2 \left(1 + 2 b_2 \eta\right) \\
k_5^* &= \alpha_m b_1^3 b_2^2 \eta
\end{aligned}
\tag{42}
$$

For the inversion it is useful to employ the formula

$$
\mathcal{L}^{-1}\left[s^{-n} \cdot e^{\frac{a}{s}}\right] = \left(\frac{t}{a}\right)^{\frac{n-1}{2}} I_o\left(2\sqrt{at}\right) \tag{43}
$$

for integer $n \geqq 1$. For $n = 0$, the formula is made valid by adding the term $\delta(t)$ to the right side; this is the Dirac delta function. In order to express the result in terms of modified Bessel function of first kind order zero and one only, we use the following recursion formula

$$
I_{n+1}(x) = I_{n-1}(x) - (2n/x)\, I_n(x) \tag{44}
$$

for modified Bessel functions (Cf.W. G. Bickley [5], Number (20a)).

It is also found convenient to introduce the function

$$
h(\ell, n; t) = e^{-bt} (a/t)^{\ell} \; I_n(2\sqrt{at}) \tag{45}
$$

for which the following differention formulas are valid.

$$
\begin{aligned}
\frac{dh(\ell, n;t)}{dt} &= h(\ell + 1/2, n-1;t) - (b + \ell/t + n/2t)\, h(\ell, n;t) \\
&= h(\ell + 1/2, n+1;t) - (b + \ell/t - n/2t)\, h(\ell, n;t)
\end{aligned}
\tag{46}
$$

In order to invert (39) consider the following function

$$\bar{f}(\eta,s) = e^{\frac{a}{s}} \sum_{n=0}^{4} k_n s^{-n} \quad ; \quad F(\eta,t) = \mathcal{L}^{-1}\left[\bar{f}(\eta,s)\right] \tag{47}$$

using the formulas,

$$\mathcal{L}^{-1}\left[e^{\frac{a}{s}}\right] = \delta(t) + \sqrt{\frac{a}{t}}\, I_1\left(2\sqrt{at}\right) \tag{48}$$

and

$$\mathcal{L}^{-1}\left[s^{-n} \cdot e^{\frac{a}{s}}\right] = \left(\frac{t}{a}\right)^{\frac{n-1}{2}} I_{n-1}\left(2\sqrt{at}\right) \qquad n \geqq 1 \tag{49}$$

yields,

$$F(\eta,t) = k_0\, \delta(t) + \sum_{n=0}^{4} k_n \left(\frac{t}{a}\right)^{\frac{n-1}{2}} I_{n-1}\left(2\sqrt{at}\right) \tag{50}$$

Next using the recursion formula (44), (50) may be expressed as

$$F(\eta,t) = k_0\, \delta(t) + m_0\, I_0(2\sqrt{at}) + m_1 \sqrt{\frac{t}{a}}\, I_1\left(2\sqrt{at}\right) \tag{51}$$

where

$$\begin{cases} m_0 = k_1 + (t/a)\left(k_3 - \dfrac{2k_4}{a}\right) \\ m_1 = \eta\, \alpha_f\, (a/t) + k_2 + 2\dfrac{k_4}{a^2} - \dfrac{k_3}{a} + k_4(t/a) \end{cases} \tag{52}$$

Thus we have

$$\mathcal{L}^{-1}\left[e^{\frac{a}{s+b_2}} \sum_{n=0}^{4} k_n (s+b_2)^{-n}\right] = e^{-b_2 t}\Big[\delta(t) + m_0\, I_0(2\sqrt{at}) + \tag{53}$$

$$\sqrt{\frac{t}{a}}\, m_1\, I_1(2\sqrt{at})\Big]$$

Now let

$$\bar{G}(\eta, s+b_2) = s \cdot \bar{f}(\eta, s+b_2) \tag{54}$$

Thus

$$\begin{aligned} G(\eta,t) &= \mathcal{L}^{-1}\left[\bar{G}(\eta, s+b_2)\right] = \mathcal{L}^{-1}\left[s \cdot \bar{f}(\eta, s+b_2)\right] \\ &= \mathcal{L}^{-1}\left[F(0^+)\right] + \mathcal{L}^{-1}\left[s\,\bar{f}(\eta, s+b_2) - F(0^+)\right] \\ &= k_1\,\delta(t) + \frac{d}{dt}F(\eta, t) \end{aligned} \tag{55}$$

To perform the indicated differentiation, (46) is used; thus

$$G(\eta,t) = e^{-b_2 t}\left[\alpha_f \eta\,\delta'(t) + \alpha_f \eta(2b_1 - b_2)\,\delta(t) + C_0 I_0(2\sqrt{at}) + C_1 I_1(2\sqrt{at})\right] \tag{56}$$

where

$$C_0 = \frac{b_1 b_2 \eta^2 \alpha_f}{t} + b_1\eta\left[\alpha_f(b_1 - 2b_2) + \alpha_m b_2\right] + \alpha_m b_1 t\left(b_1 - 2b_2 + \frac{2}{\eta}\right) \tag{57}$$

$$\begin{aligned} C_1 = &-(\alpha_f / b_1 b_2)(a/t)^{3/2} + \alpha_f \eta(2b_1 - b_2)(a/t)^{1/2} + \\ &\left(2\alpha_m b_1^2 b_2 \eta - \alpha_m b_1^2 - \alpha_m b_1 b_2^2 \eta - \alpha_f b_1^2 b_2 \eta - 2\alpha_m b_1/\eta + 2\alpha_m b_1 b_2\right)(t/a)^{1/2} - \\ &\left(\alpha_m b_1^3 b_2^2 \eta\right)(t/a)^{3/2} \end{aligned}$$

It is now easy to obtain the inversion of (39), which may be written as

$$\bar{\theta}_1(\eta,s) = e^{-\eta b_1} \cdot e^{-\eta s} \cdot \bar{G}(\eta, s+b_2) \tag{58}$$

Thus

$$\begin{aligned} \theta_1(\eta,t) &= 0 & \text{for} \quad t < \eta \\ \theta_1(\eta,t) &= e^{-\eta b_1}\, G(\eta, t-\eta) & \text{for} \quad t \geqq \eta \end{aligned} \tag{59}$$

proceeding in a similar way, consider the function

$$\bar{f}^*(\eta,s) = e^{\frac{a}{s}} \sum_{n=1}^{5} k_n^* \cdot s^{-n} \quad ; \quad F^*(\eta,t) = \mathcal{L}^{-1}\left[\bar{f}^*(\eta,s)\right] \tag{60}$$

Thus

$$F^*(\eta,t) = \sum_{n=1}^{5} k_n^* \left(\frac{t}{a}\right)^{\frac{n-1}{2}} I_{n-1}\left(2\sqrt{at}\right) \tag{61}$$

Expanding the sum in (61) and using the recursion formula (44) yields

$$F^*(\eta,t) = m_0^*\, I_0\left(2\sqrt{at}\right) + \sqrt{\frac{t}{a}}\, m_1^*\, I_1\left(2\sqrt{at}\right) \tag{62}$$

where

$$\begin{aligned} m_0^* &= k_1^* + \left(\frac{t}{a}\right)\left(k_3^* - \frac{2k_4^*}{a} - \frac{6k_5^*}{a^2}\right) + \left(\frac{t}{a}\right)^2 k_5^* \\ m_1^* &= \left(k_2^* - \frac{k_3^*}{a} + \frac{2k_4^*}{a^2} + \frac{6k_5^*}{a^3}\right) + \left(\frac{t}{a}\right)\left(k_4^* + \frac{2k_5^*}{a}\right) \end{aligned} \tag{63}$$

Again let

$$\bar{G}^*(\eta, s+b_2) = s\,\bar{f}^*(\eta, s+b_2) \tag{64}$$

Thus

$$
\begin{aligned}
G^*(\eta,t) &= \mathcal{L}^{-1}\left[\bar{G}^*(\eta, s+b_2)\right] = \mathcal{L}^{-1}\left[s\bar{f}^*(\eta, s+b_2)\right] \\
&= \mathcal{L}^{-1}\left[F^*(\eta, 0^+)\right] + \mathcal{L}^{-1}\left[s\bar{f}^*(\eta, s+b_2) - F(\eta, 0^+)\right] \qquad (65) \\
&= k_1^*\,\delta(t) + \frac{d}{dt} F^*(\eta, t)
\end{aligned}
$$

The indicated differentiation in (65) is obtained using (46). Thus

$$
G^*(\eta,t) = e^{-b_2 t}\left[\alpha_f b_2 \eta\,\delta(t) + C_o^* I_o(2\sqrt{at}) + C_1^* I_1(2\sqrt{at})\right] \qquad (66)
$$

where

$$
\begin{aligned}
C_o^* = {} & \left(\alpha_m b_2 + 2b_1 b_2 \alpha_f \eta - \alpha_f b_2^2 \eta\right) + (t/a)\,\alpha_m b_1 b_2 \,\cdot \\
& \cdot \left(2b_1 b_2 \eta + 5b_1 - b_2^2 \eta + 2b_2 + 8/\eta - b_1 b_2 \eta \alpha_f/\alpha_m\right) - \\
& (t/a)^2 \alpha_m b_1^3 b_2^3 \eta \qquad (67)
\end{aligned}
$$

$$
\begin{aligned}
C_1^* = {} & \alpha_f b_2 \eta \left(\frac{a}{t}\right)^{1/2} + \left[\alpha_m \left(b_1 b_2^2 \eta - b_1/\eta - 6b_1 b_2/\eta - 2b_2/\eta - 8/\eta^2 + 2b_1^2 b_2\right)\right] \\
& + \alpha_f b_1 b_2 \eta \left(b_1 - 2b_2 + 1/\eta\right)(t/a)^{1/2} \\
& + \left(\alpha_m b_1^2 b_2^2 \eta\right)\left(b_1 - 2b_2 - 3/\eta\right)(t/a)^{3/2}
\end{aligned}
$$

It is again easy to obtain the inversion of (41), which may be written as

$$
\bar{\bar{T}}_1(\eta, s) = e^{-\eta b_1} \cdot e^{-\eta s} \cdot \bar{G}^*(\eta, s+b_2) \qquad (68)
$$

Thus

$$T_1(\eta, t) = 0 \qquad \text{for} \qquad t < \eta$$

$$T_1(\eta, t) = e^{-b_1\eta} \cdot G^*(\eta, t-\eta) \qquad \text{for} \qquad t \geqq \eta \tag{69}$$

Although the inversion of $\bar{\theta}_1(\eta, s)$ and $\overline{T}_1(\eta, s)$ were performed in as compact a manner as possible, it is desirable to rewrite $\theta_1(\eta, t)$ and $T_1(\eta, t)$ as

$$\theta_1(\eta, t) = T_1(\eta, t) = 0 \qquad \text{for} \qquad t < \eta \tag{70}$$

$$\theta_1(\eta, t) = e^{-b_1\eta} \cdot e^{-b_2\tau} \Big[\alpha_f \eta \delta'(\tau) + \alpha_f \eta (2b_1 - b_2)\delta(\tau) +$$

$$C_0 I_0(2\sqrt{a\tau}) + C_1 I_1(2\sqrt{a\tau})\Big] \qquad \text{for} \qquad t \geqq \eta \tag{71}$$

and

$$T_1(\eta, t) = e^{-\eta b_1} \cdot e^{-b_2\tau}\Big[\alpha_f \eta b_2 \delta(\tau) + C_0^* I_0(2\sqrt{a\tau}) + C_1^* I_1(2\sqrt{a\tau})\Big] \tag{72}$$

$$\text{for} \qquad t \geqq \eta$$

It is of interest for future discussion to obtain the limiting values of $\theta_1(\eta, t)$ and $T_1(\eta, t)$ as $\tau \rightarrow 0^+$

$$\lim_{\tau \to 0^+} \theta_1(\eta, t) = e^{-b_1 \eta} \lim_{\tau \to 0^+} \left[C_0 I_0(2\sqrt{a\tau}) + C_1 I_1(2\sqrt{a\tau}) \right]$$

$$= e^{-b_1\eta} \lim_{\tau \to 0^+} \left\{ \left[\frac{b_1 b_2 \eta^2 \alpha_f}{\tau} + b_1 \eta \left[\alpha_f (b_1 - 2b_2) + \alpha_m b_2 \right] \right] (1 + a\tau + \cdots) \right.$$

$$\left. + \left[\frac{-\alpha_f a^{3/2}}{b_1 b_2 \tau^{3/2}} + \frac{\alpha_f \eta (2b_1 - b_2)\sqrt{a}}{\sqrt{\tau}} \right] \left[\sqrt{a\tau} - \frac{(a\tau)^{3/2}}{2} + \cdots \right] \right\}$$

Thus

$$\lim_{\tau \to 0^+} \theta_1(\eta, t) = e^{-b_1\eta} \cdot \alpha_f b_1 b_2 \eta \left[\frac{3}{2} b_1 b_2 \eta^2 + \eta (2b_1 - b_2) + \frac{b_1}{b_2} - 2 + \frac{\alpha_m}{\alpha_f} \right] \tag{73}$$

And similarly

$$\lim_{\tau \to 0^+} T_1(\eta, t) = e^{-b_1\eta} \cdot \lim_{\tau \to 0^+} \left[C_0^* I_0(2\sqrt{a\tau}) + C_1^* I_1(2\sqrt{a\tau}) \right]$$

$$= e^{-b_1\eta} \left[\alpha_m b_2 + 2b_1 b_2 \alpha_f \eta - \alpha_f b_2^2 \eta + \alpha_f b_2^2 b_1 \eta^2 \right] \tag{74}$$

Since the solutions obtained in (71) and (72) are complicated, they are organized so that the character of the solution can be seen as clearly as possible. Equation (70) shows that there is no disturbance ahead of the wave front. It is also noted that at the wave front ($\tau = t - \eta = 0$) θ_1 and T_1 are singular, both containing delta functions and $\theta_1(\eta, t)$ being dominated by the derivative of the delta function. Thus $\theta_1(\eta, t)$ is not of order unity and $\epsilon^2 \theta_1$ is not small in comparison to $\theta_0(\eta, t)$ at the wave front.

Although it is evident that the labor of obtaining higher order corrections (that is, higher order terms in the perturbation expansion) would be most formidable, it should be equally evident that such additional corrections would have a nature similar to that of the corrections already obtained; namely, they would exhibit discontinuous behavior at the wave front and would fail completely ahead of the wave front (that is, they would be consistent in predicting identically zero temperature). However, it is to be expected that they would possess calculable limiting values, from the left, as the wave front is approached. In effect, we thus have what appears to be a perfectly valid way of determining the behavior behind the wave front, but one which fails at, and ahead of the wave front. The nature of the failure of our original perturbation expansion is characteristic of singular perturbation problems [9], and we are therefore led to seek a singular perturbation expansion which will resolve the difficulties which have been encountered. This is done in the following subsection.

C. SINGULAR PERTURBATION ANALYSIS

We have seen that the perturbation solution represented by the series (12) and (13) involve no difficulties, except those of analytical complexity, for any $\tau > 0$, that is, at any point behind the wave front. However, this solution is useless for points ahead of the front $\tau < 0$, since it predicts zero temperature for fluid and solid at all points ahead of the wave front; physical considerations indicate that

diffusion through the wave front will result in non-zero temperatures ahead of the front. A mathematical indication of the difficulty may be seen from the fact that $\epsilon^2 \theta_1$ is not small at the wave front, and presumably the same thing could be shown for $\epsilon^4 \theta_2$, etc. Accordingly, we now look for another solution which will be valid ahead of the wave front. To do this it is convenient to introduce the new variable

$$\xi = (\eta - t) / \epsilon = x - Vt \tag{75}$$

and we now seek new solutions $\theta(\xi, t)$, $T(\xi, t)$ which will be valid for $\xi > 0$.

Making the change of independent variable from (η, t) to (ξ, t) provides the differential equations

$$\alpha_f \frac{\partial^2 \theta}{\partial \xi^2} = \frac{\partial \theta}{\partial t} + b_1 (\theta - T) \tag{76}$$

$$\epsilon \alpha_m \frac{\partial^2 T}{\partial \xi^2} = -\frac{\partial T}{\partial \xi} + \epsilon \frac{\partial T}{\partial t} - \epsilon b_2 (\theta - T) \tag{77}$$

Note that the transformation (75) changes our viewpoint so that, in effect, we are now moving to the right with the front so that the fluid appears to be stationary and the pipe appears to be moving to the left with velocity V.

Note that in contradistinction with the situation dealt with previously, now ϵ (to the first power) appears in the governing equations.

Accordingly ϵ , not ϵ^2, appears as the natural parameter with respect to which we hope to construct a perturbation expansion. Thus we deal with

$$\theta(\xi,t) = \theta_0(\xi,t) + \epsilon\,\theta_1(\xi,t) + \epsilon^2\,\theta_2(\xi,t) + \cdots \qquad (78)$$

$$T(\xi,t) = T_0(\xi,t) + \epsilon\,T_1(\xi,t) + \epsilon^2 T_2(\xi,t) + \cdots \qquad (79)$$

(Later when we "match" these new solutions to the solutions discussed in the previous subsection, we will add an asterisk to distinguish the new solutions.

The conditions that all these new functions must satisfy are

$$\theta_i \longrightarrow 0 \quad , \quad T_i \longrightarrow 0 \qquad \text{as} \qquad \xi \longrightarrow \infty \qquad (80)$$

$$\theta_i = T_i = 0 \qquad \text{for} \qquad t = 0 \quad . \qquad (81)$$

for i = 0, 1, 2, ...

Additionally, they must satisfy appropriate conditions at the wave front. Actual selection, or determinations, of these conditions for $\xi = 0$ will be made in the next subsection where the matching process will be treated.

Substituting (78) and (79) into (76) and (77) comparing powers of ϵ , gives rise to the following sequence of equations. For the fluid, we will have

$$\alpha_f \theta_0'' = \dot{\theta}_0 + b_1 (\theta_0 - T_0) \tag{82}$$

$$\alpha_f \theta_1'' = \dot{\theta}_1 + b_1 (\theta_1 - T_1) \tag{83}$$

$$\alpha_f \theta_2'' = \dot{\theta}_2 + b_1 (\theta_2 - T_2) \tag{84}$$

$$\cdots$$

and for the solid we obtain

$$T_0' = 0 \tag{85}$$

$$\alpha_m T_0'' = -T_1' + \dot{T}_0 - b_2 (\theta_0 - T_0) \tag{86}$$

$$\alpha_m T_1'' = -T_2' + \dot{T}_1 - b_2 (\theta_1 - T_1) \tag{87}$$

$$\cdots$$

(where the prime denotes partial derivative with respect to ξ and dot denotes partial derivative with respect to time).

Condition (80) and equation (85) show that

$$T_0 \equiv 0 \tag{88}$$

Now consider the first non-trivial equation in the sequence, namely (82). This is a classical diffusion equation, the solution of which, subject to (80), is

$$\theta_0(\xi, t) = A_0 e^{-b_1 t} \cdot \operatorname{erfc}\left(\xi / 2\sqrt{\alpha_f t}\right) \tag{89}$$

The constant A_o is available to satisfy a boundary condition for $\xi = 0$. The next equation to be considered is (86) which in view of (88) may be written as

$$T_1' = - b_2 \theta_o \tag{90}$$

Together with conditions (80) and (81), we may write down the solution of (90) as

$$T_1(\xi, t) = A_o b_2 e^{-b_1 t} \int_{\xi}^{\infty} \operatorname{erfc}\left(\xi / 2\sqrt{\alpha_f t}\right) d\xi \tag{91}$$

Note that no new constant of integration is required in the last step.

The next equation is (83) which is similar to (82) except that T_1, given by (91), is involved. The solution could be obtained but we will not do more than remark that another constant of integration will be introduced. Then (87) provides T_2; no new constant is introduced. Next (84) provides a means of determining θ_2; another constant of integration is involved.

Briefly the solution for θ_i ($i = 0, 1, 2, \ldots$) involves a constant A_i, which is actually a function of time t. Since the equations for T_i are first order, their solutions do not require the introduction of arbitrary constants or functions, equation (80) and (81) providing the necessary conditions.

No difficulties are encountered (except those of increasing analytical complexity) in continuing this process and we expect

convergent series (78) and (79). However, as yet we have made no move to relate these expansions to any conditions at the left which might serve for the establishment of the as yet undetermined functions $A_i(t)$. We proceed to do this in what follows immediately.

D. MATCHING OF SOLUTIONS

We now employ an asterisk to distinguish the solutions (θ_i^* and T_i^*) of the immediately preceding subsection, valid ahead of the front (i.e., for $\xi > 0$), from the solutions (θ_i and T_i) of the earlier subsection, valid behind the front (i.e., for $\tau > 0$). We will employ the following notations.

$$\lim_{x \to Vt^-} \theta_i(x,t) = \bar{\theta}_i(t) = \bar{\theta}_i \qquad (92)$$

$$\lim_{x \to Vt^-} T_i(x,t) = \bar{T}_i(t) = \bar{T}_i \qquad (93)$$

$$\left[\theta_i^*(\xi,t)\right]_{\xi=0} = \bar{\theta}_i^*(t) = \bar{\theta}_i^* \qquad (94)$$

$$\left[T_i^*(\xi,t)\right]_{\xi=0} = \bar{T}_i^*(t) = \bar{T}_i^* \qquad (95)$$

(Note that here an overbar does <u>not</u> indicate a Laplace transform.)

Examples are

$$\bar{\theta}_0 = e^{-b_1 t} \qquad (96)$$

$$\overline{T}_o = 0 \tag{97}$$

$$\overline{\theta}_1 = e^{-b_1 t} \cdot \alpha_f b_1 b_2 t \left[\frac{3}{2} b_1 b_2 t^2 + (2b_1 - b_2) t + \frac{b_1}{b_2} - 2 + \frac{\alpha_m}{\alpha_f} \right] \tag{98}$$

$$\overline{T}_1 = e^{-b_1 t} \left(\alpha_m b_2 + 2 b_1 b_2 \alpha_f t - \alpha_f b_2^2 t + \alpha_f b_2^2 b_1 t^2 \right) \tag{99}$$

$$\overline{T}_o^{\,*} = 0 \tag{100}$$

$$\overline{\theta}_o^{\,*} = A_o\, e^{-b_1 t} \tag{101}$$

$$\overline{T}_1^{\,*} = A_o b_2\, e^{-b_1 t} \int_0^{\infty} erfc \left(\zeta / 2\sqrt{\alpha_f t} \right) d\zeta \tag{102}$$

. . .

The functions θ_i^* and T_i^* contain as yet undetermined functions of t, namely $A_o = A_o(t)$, $A_1 = A_1(t)$, etc., and we must be able to evaluate these systematically. Our procedure is to evaluate so that the perturbation sequences to the left and right of the front, truncated so as to include only terms containing ϵ to an arbitrary even power (and lower powers) will match at the interface.

The first matching is to require

$$\overline{\theta}_o = \overline{\theta}_o^{\,*} \tag{103}$$

$$\overline{T}_o = \overline{T}_o^{\,*} \tag{104}$$

Equation (104) is automatically satisfied. Equation (103) provides a condition which can be used to determine the constant A_o. We find

$A_o = A_o(t) = 1$. Thus, our first (matched) approximation is given by (29), (31), (89), and (91) with $A_o = 1$ in the latter two.

The second matching is

$$\bar{\theta}_0 + \epsilon^2 \bar{\theta}_1 = \bar{\theta}_0^* + \epsilon \bar{\theta}_1^* + \epsilon^2 \bar{\theta}_2^* \tag{105}$$

$$\bar{T}_0 + \epsilon^2 \bar{T}_1 = \bar{T}_0^* + \epsilon \bar{T}_1^* + \epsilon^2 \bar{T}_2^* \tag{106}$$

The functions θ_1^*, θ_2^*, T_1^*, and T_2^* have introduced two additional unknown functions $A_1(t)$ and $A_2(t)$. Equations (105) and (106) provide conditions for their determination.

Similarly the next matching is

$$\bar{\theta}_0 + \epsilon^2 \bar{\theta}_1 + \epsilon^4 \bar{\theta}_2 = \bar{\theta}_0^* + \epsilon \bar{\theta}_1^* + \epsilon^2 \bar{\theta}_2^* + \epsilon^3 \bar{\theta}_3^* + \epsilon^4 \bar{\theta}_4^* \tag{107}$$

$$\bar{T}_0 + \epsilon^2 \bar{T}_1 + \epsilon^4 \bar{T}_2 = \bar{T}_0^* + \epsilon \bar{T}_1^* + \epsilon^2 \bar{T}_2^* + \epsilon^3 \bar{T}_3^* + \epsilon^4 \bar{T}_4^* \tag{108}$$

in which two additional functions, $A_3(t)$ and $A_4(t)$ appear and which afford means for their determination. Proceeding in this manner, all functions $A_i(t)$ may be determined. At each stage we have matched solutions to the same degree in ϵ, on either side of the front.

By this procedure, eventually (1) and (4) are satisfied. Moreover, at any stage, the necessary coefficients $A_i(t)$ are calculable so that the approximation affords truncated series, available both to the left and to the right of the front, with matching at the front. Thus it appears

that we have succeeded in describing a process which provides two sets of series expansions, one valid behind the front and the other valid ahead of the front. (A " et" consists of a solution θ for the fluid and a solution T for the solid.) The labor of constructing the successive functions that occur in these series is considerable and we have actually obtained only a few of them. We have actually made the "first matching" and could, with additional labor, determine sufficient additional functions to accomplish second, and even third matchings. However, the writer confesses to a feeling of uncertainty in this entire matter of constructing the ordinary and the singular perturbation expansions and the matter of performing a matching where it is necessary to join them. Accordingly he suggests that it may be fruitful to consider this matter further, employing the insights which might be provided by a more sophisticated mathematical treatment.

E. A DIFFERENT SOLUTION AHEAD OF TEMPERATURE FRONT

The solution given in this subsection may be of some interest although it involves a certain basic inconsistency. In this solution we are concerned with the temperature variation ahead of the front and are able to consider both radial and axial variation in the fluid, as well as both axial and radial diffusion in the fluid. However, we assume that the temperature of the solid is uniformly zero and that there is perfect heat transfer through the interface; that is, we

consider an isothermal boundary condition for the fluid. (It would not alter the problem profoundly if a finite surface coefficient of heat transfer were assumed.) Thus, this problem is not a conjugated problem. For a left end boundary condition we assume that the fluid temperature is a constant, independent of radius, which depends on the location of the front. Accordingly, the degree of approximation and inconsistency in this solution is considerably greater than is to be found in other problems treated herein. However, the results may be useful in considering the temperature distribution ahead of the front at quite small values of time, before significant radial variation has built up in the front.

We will consider equation (II-1) and (II-2) with isotropic thermal conductivities and subject them to a change of variable

$$\xi = x - Vt \qquad ; \qquad t = t \tag{109}$$

Then equation (II-1) reduces to

$$\frac{\partial^2 \theta}{\partial \xi^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} + \frac{\partial^2 \theta}{\partial r^2} = \frac{1}{\alpha_f}\frac{\partial \theta}{\partial t} \tag{110}$$

Similarly equation (II-2) reduces to

$$\frac{\partial^2 T}{\partial \xi^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} + \frac{V}{\alpha_m}\frac{\partial T}{\partial \xi} = \frac{1}{\alpha_m}\frac{\partial T}{\partial t} \tag{111}$$

We note that the effect of the change of variable, (109), is to change our view point as follows; the fluid remains stationary and the pipe

wall moves to the left with the same velocity V of the fluid. If the velocity V is large then we state the following assumption

Assume that for $\xi > 0$ the pipe wall is essentially at zero temperature.

Hence we will concentrate our attention on equation (109) and find the solution when equation (109) is subjected to the following conditions

$$\theta = 1 \qquad \xi = 0 \quad , \quad 0 \leq r \leq a \quad , \quad t \geq 0 \tag{112}$$

$$\theta = 0 \qquad \xi \to \infty \quad , \quad 0 \leq r \leq a \quad , \quad t \geq 0 \tag{113}$$

$$\theta = 0 \qquad 0 \leq \xi < \infty \quad , \quad r = a \quad , \quad t \geq 0 \tag{114}$$

$$\frac{\partial \theta}{\partial r} = 0 \qquad 0 \leq \xi < \infty \quad , \quad r = 0 \quad , \quad t \geq 0 \tag{115}$$

$$\theta = 0 \qquad 0 \leq \xi < \infty \quad , \quad 0 \leq r \leq a \quad , \quad t = 0 \tag{116}$$

(Note that $\xi = 0$ implies $x = Vt$, and $\xi \to \infty$ implies $x \to \infty$)

Assume the solution to be

$$\theta = \sum_{n=1}^{\infty} A_n \, \Phi_n(\xi, t) \, J_0(\lambda_n r) \tag{117}$$

condition (114) yields

$$J_0(\lambda_n a) = 0 \tag{118}$$

and condition (115) is already satisfied.

Next, taking Laplace transforms of (110) and (117), we find

$$\frac{\partial^2 \bar{\Phi}_n}{\partial \xi^2} - \left(\lambda_n^2 + \frac{s}{\alpha_f}\right) \bar{\Phi}_n = 0 \tag{119}$$

A solution of (119) which satisfies (113) is

$$\bar{\Phi}_n = B_n \exp\left(-\xi \sqrt{\lambda_n^2 + \frac{s}{\alpha_f}}\right) \tag{120}$$

Thus, writing $C_n = A_n B_n$, we have

$$\bar{\theta} = \sum_{n=1}^{\infty} C_n J_0(\lambda_n r) \exp\left[-\xi \sqrt{\lambda_n^2 + \frac{s}{\alpha_f}}\right] \tag{121}$$

Condition (112) then yields

$$\frac{1}{s} = \sum_{n=1}^{\infty} C_n J_0(\lambda_n r) \tag{122}$$

Using the orthogonality of Bessel functions, viz.

$$\int_0^a J_0(\lambda_n r) J_0(\lambda_m r)\, r\, dr = \frac{a^2}{2} J_1^2(\lambda_n a) \tag{123}$$

if n = m and zero if n ≠ m, we find,

$$C_n = \frac{1}{s} \cdot \frac{2}{\lambda_n a} \cdot \frac{1}{J_1(\lambda_n a)} \tag{124}$$

Hence

$$\bar{\theta} = 2\sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{(\lambda_n a) J_1(\lambda_n a)} \left[\frac{1}{s} \exp\left(-\xi \sqrt{\lambda_n^2 + \frac{s}{\alpha_f}}\right)\right] \tag{125}$$

Inverting (125) we find

$$\theta(\xi,r,t) = \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{(\lambda_n a) J_1(\lambda_n a)} \left[e^{-\lambda_n \xi} \cdot \mathrm{erfc}\left(\frac{\xi - 2\alpha_f \lambda_n t}{2\sqrt{\alpha_f t}}\right) + e^{\lambda_n \xi} \cdot \mathrm{erfc}\left(\frac{\xi + 2\alpha_f \lambda_n t}{2\sqrt{\alpha_f t}}\right)\right] \quad (126)$$

(Note ; that solution (126) is for the case when the wave front is maintained at unity.) It is now possible to use Duhamel's superposition integral when the condition at $\xi = 0$ is

$$\theta_0(0,r,t) = \lim_{\tau \to 0^+} \theta_0(\eta, t) = e^{-b_1 t} \quad (127)$$

which is the simplest estimate we have, Cf. (96). Thus, we obtain

$$\theta(\xi,r,t) = \int_0^t e^{-b_1 s} \cdot \frac{\partial}{\partial z}\Big[\theta(\xi,r,z)\Big]_{z=t-s} \cdot ds \quad (128)$$

where the term in brackets may be obtained by differentiating (126).

Obviously, this solution is for cylindrical geometry, but the case of slab geometry differs only in detail.

V. CASE WHERE FLUID TEMPERATURE IS AVERAGED AND AXIAL CONDUCTION IN CONDUIT IS NEGLECTED

A. SLAB GEOMETRY

1. Development of Solution

In this section equations (II-19) and (II-20), together with the appropriate boundary and initial conditions will be further developed. For convenience of reference these equations are rewritten below.

$$\frac{\partial^2 \theta}{\partial x^2} - \frac{V}{\alpha_f}\frac{\partial \theta}{\partial x} = \frac{1}{\alpha_f}\frac{\partial \theta}{\partial t} + \frac{h}{a K_f}(\theta - T) \tag{1}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha_m}\frac{\partial T}{\partial t} \tag{2}$$

$$\theta = 0 \qquad 0 \leqq x < \infty \quad , \quad t = 0 \tag{3}$$

$$\theta = 1 \qquad x = 0 \quad , \quad t \geqq 0 \tag{4}$$

$$\theta \rightarrow 0 \qquad x \rightarrow \infty \quad , \quad t \geqq 0 \tag{5}$$

$$T = 0 \qquad 0 \leqq y \leqq b \quad , \quad t = 0 \tag{6}$$

$$\frac{\partial T}{\partial y} = 0 \qquad y = b \quad , \quad t \geqq 0 \tag{7}$$

$$-K_m \frac{\partial T}{\partial y} = h(\theta - T) \qquad y = 0 \quad , \quad t \geqq 0 \tag{8}$$

Note, that $\theta = \theta$ (x, t) appearing here is the average fluid temperature as defined in (II-18); we have omitted the asterisk uniformly throughout

this section. Also, $T = T(x, y, t)$ is the temperature in the duct wall; we have omitted the subscript 2 on y; in this section, the symbol y denotes y_2 in figure 1b. We also note that this analysis neglects the axial conduction in the duct wall. Finally, note that the surface heat transfer coefficient appearing above is the "equivalent" film resistance coefficient $\tilde{h}$ discussed following (II-17); we omit writing the tilde in (1) and (8) above and in what follows.

In order to solve (1) and (2), Laplace transformation is used. Equations (1), (3), (4) and (5) become

$$\frac{\partial^2 \bar{\theta}}{\partial x^2} - \frac{V}{\alpha_f}\frac{\partial \bar{\theta}}{\partial x} = \frac{s}{\alpha_f}\bar{\theta} + \frac{h}{a K_f}\left[\bar{\theta} - (\bar{T})_{y=0}\right] \qquad (9)$$

$$\bar{\theta} = \frac{1}{s} \qquad\qquad x = 0 \qquad (10)$$

$$\bar{\theta} \to 0 \qquad\qquad x \to \infty \qquad (11)$$

Similarly equations (2), (6), (7) and (8) become

$$\frac{\partial^2 \bar{T}}{\partial y^2} = \frac{s}{\alpha_m}\bar{T} \qquad (12)$$

$$-K_m \frac{\partial \bar{T}}{\partial y} = h(\bar{\theta} - \bar{T}) \qquad 0 \leqq x < \infty \;, \quad y = 0 \qquad (13)$$

$$\frac{\partial \bar{T}}{\partial y} = 0 \qquad 0 \leqq x < \infty \;, \quad y = b \qquad (14)$$

The solution of (12) is

$$\bar{T} = A e^{qy} + B e^{-qy} \tag{15}$$

where

$$q = \sqrt{\frac{S}{\alpha_m}} \tag{16}$$

Condition (14) permits evaluating B in terms of A, and we arrive at

$$\bar{T} = A \left[e^{qy} + e^{q(2b-y)} \right] \tag{17}$$

Since the terms in parenthesis are clearly independent of x, the imposition of condition (13) implies that A in (17) must be a function of x and s. Thus using (13) to express $\bar{\theta}$ in terms of $\overline{T}$, we find

$$\bar{\theta} = \beta(s) \cdot A(x,s) \tag{18}$$

where

$$\beta(s) = \frac{-K_m}{h} q \left(1 - e^{2qb}\right) + \left(1 + e^{2qb}\right) \tag{19}$$

Substituting from (17) and (18) into (9) yields a homogeneous second order ordinary differential equation for A(x, s). After rearranging and substitution for β (s) this becomes

$$\frac{\partial^2 A}{\partial x^2} - \frac{V}{\alpha_f} \frac{\partial A}{\partial x} = g(s) \cdot A(x,s) \qquad (20$$

where

$$g(s) = \frac{s}{\alpha_f} + \frac{1}{a K_f}\left(\frac{K_m \xi \tanh(b\xi)}{1 + \frac{K_m}{h}\xi \tanh(b\xi)}\right) \tag{21}$$

The general solution of (20) is

$$A(x,s) = C \cdot \exp\left[\frac{Vx}{2\alpha_f} - x\sqrt{\left(\frac{V}{2\alpha_f}\right)^2 + g(s)}\right] + D \cdot \exp\left[\frac{Vx}{2\alpha_f} + x\sqrt{\left(\frac{V}{2\alpha_f}\right)^2 + g(s)}\right] \tag{22}$$

where C and D are constants of integration which depend on S. The boundary conditions (10) and (11) in conjunction with (18) define the following conditions for A (x , s)

$$A(x,s) = \frac{1}{s \cdot \beta(s)} \qquad x = 0 \tag{23}$$

$$A(x,s) \longrightarrow 0 \qquad x \longrightarrow \infty \tag{24}$$

Condition (24) clearly implies that D $\equiv$ 0 whereas (25) yields

$$C = \frac{1}{s \cdot \beta(s)} \tag{25}$$

Therefore

$$A(x,s) = \frac{1}{s \cdot \beta(s)} \exp\left[\frac{Vx}{2\alpha_f} - x\sqrt{\left(\frac{V}{2\alpha_f}\right)^2 + g(s)}\right] \tag{26}$$

Substitution of A(x, s) from (26) above into (18) and (17) yields the fluid and pipe wall temperature distribution in Laplace transform form; viz.

$$\bar{\theta}(x,s) = \frac{1}{s} \cdot \exp\left[\frac{Vx}{2\alpha_f} - x\sqrt{\left(\frac{V}{2\alpha_f}\right)^2 + \frac{s}{\alpha_f} + \frac{1}{a K_f}\left(\frac{K_m q \tanh(bq)}{1 + \frac{K_m}{h} q \tanh(bq)}\right)}\right] \tag{27}$$

and

$$\bar{T}(x,y,s) = \left\{\frac{\cosh[q(b-y)]}{\cosh(bq) + \frac{K_m}{bh}(bq)\sinh(bq)}\right\} \cdot \tag{28}$$

$$\cdot \left\{\frac{1}{s}\exp\left[\frac{Vx}{2\alpha_f} - x\sqrt{\left(\frac{V}{2\alpha_f}\right)^2 + \frac{s}{\alpha_f} + \frac{1}{aK_f} \cdot \frac{K_m q \tanh(bq)}{1 + \frac{K_m}{bh}(bq)\tanh(bq)}}\right]\right\} = \bar{F}(y,s) \cdot \bar{\theta}(x,s)$$

In order to invert (27) and (28), we first consider $\bar{\theta}(x, s)$. Denoting the radicand by m we have.

$$m = \frac{\left(\frac{V}{2\alpha_f}\right)^2 + \frac{s}{\alpha_f} + (bq)\tanh(bq)\left[\frac{K_m}{bh}\left(\frac{V}{2\alpha_f}\right)^2 + \frac{K_m}{bh} \cdot \frac{s}{\alpha_f} + \frac{K_m}{abK_f}\right]}{1 + \frac{K_m}{bh}(bq)\tanh(bq)} \tag{29}$$

It will be convenient to cast this equation in another form. Write

$$\lambda = -iq = -i\sqrt{\frac{s}{\alpha_m}} \tag{30}$$

so that

$$s = -\alpha_m \lambda^2 \quad ; \quad q = i\lambda \tag{31}$$

Then (29) becomes

$$m = \frac{B_m\left[\left(\frac{V}{2\alpha_f}\right)^2 - \frac{\alpha_m}{\alpha_f}\lambda^2\right] - b\lambda \tan\lambda b\left[\left(\frac{V}{2\alpha_f}\right)^2 + \frac{B_f}{a^2} - \frac{\alpha_m}{\alpha_f}\lambda^2\right]}{B_m - b\lambda\tan(b\lambda)} \tag{32}$$

where

$$B_m = \frac{bh}{K_m} \qquad ; \qquad B_f = \frac{ah}{K_f} \tag{33}$$

are Biot numbers for the wall and for the fluid, respectively. We are concerned with finding the singularities of (27). Except for the pole at s=0, these will all be branch points corresponding to the zeros and poles of m . Considering the numerator and denominator of (32) separately, we first note that the denominator has poles for $b\lambda = (n+1/2)\pi$, n = 0, 1, 2... . These will also be poles for the numerator unless

$$\left(\frac{V}{2\alpha_f}\right)^2 + \frac{B_f}{a^2} = \frac{\alpha_m}{\alpha_f}\left(n+1/2\right)^2\frac{\pi^2}{b^2} \tag{34}$$

We may assume that this equality does not hold for any integer n for otherwise we could make an infinitesimally small change in any of the physical parameters so as to assure that the equality is violated, without making any significant change in the physical problem. Thus, the only zeros and poles of m correspond to the zeros of the numerator and denominator, respectively.

The condition for zeros of the numerator (i.e., zeros of m), is that

$$\tan(\lambda b) = f_1(\lambda) \tag{35}$$

where

$$f_1(\lambda) = \frac{B_m}{\lambda b} \cdot \frac{\left(\frac{V}{2\alpha_f}\right)^2 - \frac{\alpha_m}{\alpha_f}\lambda^2}{\left(\frac{V}{2\alpha_f}\right)^2 + \frac{B_f}{a^2} - \frac{\alpha_m}{\alpha_f}\lambda^2} \tag{36}$$

and the condition for zeros of the denominator (i. e., poles of m), is that

$$\tan(b\lambda) = f_2(\lambda) \tag{37}$$

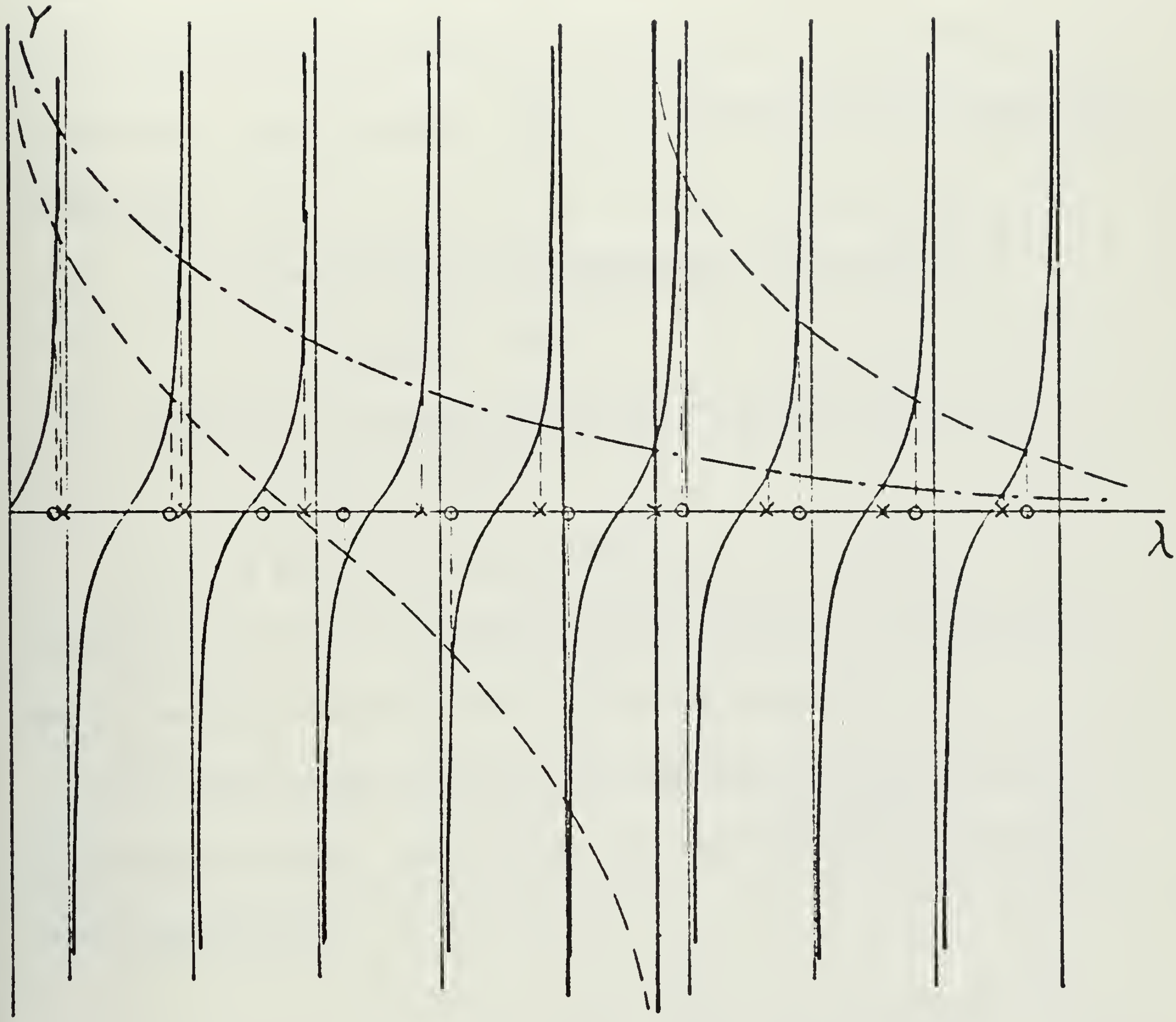
where

$$f_2(\lambda) = \frac{B_m}{b\lambda} \tag{38}$$

The real zeros of (35) and (37) may be found easily. Their location is illustrated in Figure 3. The relative position of the dashed line and the dot-dash line in Figure 3, is important. For small λ it is evident that $f_1(\lambda) < f_2(\lambda)$. Also $f_1(\lambda)$ has a vertical asymptote for a value of λ greater than that of its single zero. Thus the positions are as shown in Figure 3.

Finally, we must assume that there are no other zeros than the real zeros indicated in Figure 3. However, both (36) and (38) satisfy the hypotheses of Appendix B which shows that there are no non-real zeros.

Thus it is clear that the only singularities of m are poles; furthermore, m has only real zeros and poles, infinite in number, and alternating with each other, of which that having the smallest



—— $y = \tan(b\lambda)$

- - - - $y = f_1(\lambda)$

—·—·— $y = f_2(\lambda)$

o - zeros, x - poles

Figure 3: Poles and Zeros of $m(s)$
(Slab Geometry)

numerical value is a zero. If we designate the abcissae of the intersections denoted by 0 in Figure 3 by the symbol λ_n and those denoted by x by the symbol λ_n^*, we conclude that the singularities of (27) are:

a. branch points (corresponding to zeros of m) at

$S = -\alpha_m \lambda_n^2$, and

b. branch points (corresponding to poles of m) at

$S = -\alpha_m \lambda_n^{*2}$

c. a pole at S=0 with residue 1

Furthermore, the branch points of type a and type b alternate with each other, that closest to the origin being of type a.

Thus the hypotheses of the inversion theorem given in Appendix A hereof are satisfied. Accordingly, we may immediately write the inverse as follows

$$\theta(x,t) = 1 - \frac{1}{\pi} \exp\left(\frac{Vx}{2\alpha_f}\right) \sum_{K=1}^{\infty} \int_{a_K}^{b_K} \frac{e^{-\eta t}}{\eta} \sin\left[x\mu(\eta)\right] d\eta \qquad (39)$$

where $\mu(\eta) = \left(\left| m(-\eta)\right|\right)^{1/2}$

Similarly T(x, y, s) as given by (28) is conveniently organized in the form of the product of two Laplace transforms. Accordingly, its inverse may be easily obtained in the form of a convolution integral.

$$T(x,y,t) = 4\sum_{n=1}^{\infty} \frac{\alpha_m \lambda_n^2 \sin(\lambda_n b)}{2\lambda_n b + \sin(2\lambda_n b)} \cos\left[\lambda_n (b-y)\right] \cdot \cdot e^{-\alpha_m \lambda_n^2 t} \int_0^t e^{\alpha_m \lambda_n^2 \eta} \cdot \theta(x,\eta)\, d\eta \tag{40}$$

where λ_n are the roots of (37) and the integrand contains the function (39). We have used the formula

$$\mathcal{L}^{-1}\left[\frac{\cosh(a s^{1/2})}{\cosh(b s^{1/2}) + k(b s^{1/2}) \sinh(b s^{1/2})}\right] = 4\sum_{n=1}^{\infty} \frac{\lambda_n^2 \, e^{-\lambda_n^2 t} \sin(\lambda_n a) \cos(\lambda_n b)}{2\lambda_n b + \sin(2b\lambda_n)} \tag{41}$$

which may be derived directly or inferred from equations given in [2], examples 5-3 and 7-11.

2. Asymptotic Approximations

Thus the desired solution of the system described by equation (1) through (8) has been achieved. We will not in this work discuss computational aspects, but will now obtain some asymptotic approximations.

Equation (27) can also be written

$$\bar{\theta}(x,s) = \frac{1}{s} \exp\left[\frac{Vx}{2\alpha_f} - x\sqrt{\left(\frac{V}{2\alpha_f}\right)^2 + \frac{s}{\alpha_f} + \frac{K_m}{abK_f}\left(\chi - \chi^2/B_m + \chi^3/B_m^2 \cdots\right)}\right] \tag{42}$$

where

$$\chi = bq \tanh(bq) = \frac{sb^2}{\alpha_m} - \frac{s^2 b^4}{3\alpha_m^2} + \frac{2}{15}\frac{s^3 b^6}{\alpha_m^3} - \frac{17}{35}\frac{s^4 b^8}{\alpha_m^4} + \cdots \tag{43}$$

Consider the radicand to be expanded in ascending powers of s.

$$\bar{\theta}(x,s) = \frac{1}{s}\exp\left[\frac{Vx}{2\alpha_f} - x\sqrt{\left(\frac{V}{2\alpha_f}\right)^2 + \frac{1}{\alpha_f}\left(1 + \frac{K_m b \alpha_f}{K_f \alpha_m a}\right)s + O(s^2)}\right] \tag{44}$$

Considering small values of s, which corresponds to large t, the first asymptotic approximation is obtained by neglecting all powers of s except the zeroth power. Correspondingly

$$\bar{\theta}(x,s) = \frac{1}{s} \tag{45}$$

which yields

$$\theta(x,t) = 1 \tag{46}$$

This is evidently correct as an asymptotic expansion since for any finite x the temperature approaches unity for sufficiently large time.

A nontrivial asymptotic expansion is obtained by retaining the first power of s. Correspondingly

$$\bar{\theta}(x,s) = \frac{1}{s}\exp\left[\frac{Vx}{2\alpha_f} - x\sqrt{\left(\frac{V}{2\alpha_f}\right)^2 + \frac{s}{\alpha_f} + \frac{K_m A_m}{K_f A_f}\cdot\frac{s}{\alpha_m}}\right] \tag{47}$$

The reason for writing A_m / A_f, the area ratio, in place of b/a which naturally would appear in the last term of the radicand will appear in Section VB2.

The inversion of (47) is tabulated [29], thus

$$\theta(x,t) = \frac{1}{2}\left[\operatorname{erfc}\left(\frac{x-V^*t}{2\sqrt{\alpha_f^* t}}\right) + e^{\frac{Vx}{\alpha_f}}\operatorname{erfc}\left(\frac{x+V^*t}{2\sqrt{\alpha_f^* t}}\right)\right] \tag{48}$$

where

$$V^* = V\Big/\left(1+\rho_m c_m A_m/\rho_f c_f A_f\right) \tag{49}$$

is a retarded velocity which is identical to that obtained by Munk [24], and

$$\alpha_f^* = \alpha_f\Big/\sqrt{1+\rho_m c_m A_m/\rho_f c_f A_f} \tag{50}$$

is an effective thermal diffusivity for the fluid. This solution should be compared with that given in (VII-10) where a different set of idealizations leads to a similar result. There, the retarded velocity (VII-5) is the same as (49), but the diffusivity (VII-6) is slightly different from that given in (50).

For the corresponding solution for the conduit wall temperature we employ (48) in (40).

Retaining the second power of s would yield a useful improved asymptotic approximation if one could find a simple explicit inversion formula. Unfortunately nothing simpler than the general inversion given in Appendix A has come to light, and this has provided the complete solution given in (39) and (40).

B. CYLINDRICAL GEOMETRY

1. Development of Solution

A similar treatment to that given in Section VA is employed. Only the significantly different details will be treated explicitly. Equation (2) will be replaced by

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha_m}\frac{\partial T}{\partial t} \tag{51}$$

with the corresponding conditions

$$T = 0 \qquad a \leqq r \leqq b \quad , \quad t = 0 \tag{52}$$

$$\frac{\partial T}{\partial r} = 0 \qquad r = b \quad , \quad t \geqq 0 \tag{53}$$

$$-K_m \frac{\partial T}{\partial r} = h\left(\theta - T\right) \qquad r = a \quad , \quad t \geqq 0 \tag{54}$$

replacing (6), (7), and (8). Remarks similar to those following (8) are applicable here.

As previously, Laplace transformations are made with respect to t. The solution of the transformed version of (51) is

$$\bar{T} = A I_0(q r) + B K_0(q r) \tag{55}$$

where

$$q = \sqrt{s/\alpha_m} \tag{56}$$

is the same as previously. Here I_0 and K_0 denote modified Bessel functions.

Proceeding as in VA1, one arrives at equations similar to (27) and (28) except that the function $\tanh$ (qb) appearing therein is to be replaced by another function which we denote

$$\operatorname{banh}(x, \beta) = \frac{I_1(x) K_1(\beta x) - I_1(\beta x) K_1(x)}{I_0(\beta x) K_1(x) + K_0(\beta x) I_1(x)} \tag{57}$$

(The initial b denotes "Bessel." The remainder (anh) is intended to indicate a similarity to the function $\tanh(x)$.)

Corresponding to (29), we have in this case, the equation.

$$m = \frac{\left(\frac{V}{2\alpha_f}\right)^2 + \frac{S}{\alpha_f} + (bq)\operatorname{banh}(bq, a/b)\left[\frac{K_m}{bh}\left(\frac{V}{2\alpha_f}\right)^2 + \frac{K_m}{bh}\frac{S}{\alpha_f} + \frac{2K_m}{abK_f}\right]}{1 + \frac{K_m}{bh}(bq)\operatorname{banh}(bq, a/b)} \tag{58}$$

Corresponding to the well known relation

$$i \tanh(x) = \tan(ix) \tag{59}$$

it is not difficult to show that

$$i \operatorname{banh}(x, \beta) = \operatorname{ban}(ix, \beta) \tag{60}$$

where

$$\operatorname{ban}(x, \beta) = \frac{J_1(x) Y_1(\beta x) - Y_1(x) J_1(\beta x)}{J_1(x) Y_0(\beta x) - Y_1(x) J_0(\beta x)} \tag{61}$$

Thus corresponding to (32), upon defining

$$\lambda = -i q \tag{62}$$

we obtain

$$m = \frac{B_m\left[\left(\frac{V}{2\alpha_f}\right)^2 - \frac{\alpha_m}{\alpha_f}\lambda^2\right] - (b\lambda)\, \mathrm{ban}(b\lambda, a/\lambda)\ \left[\left(\frac{V}{2\alpha_f}\right)^2 - \frac{\alpha_m}{\alpha_f}\lambda^2 + \frac{B_f^*}{a^2}\right]}{B_m - (b\lambda)\, \mathrm{ban}(b\lambda, a/b)} \tag{63}$$

where

$$B_m = \frac{b h}{K_m} \qquad ; \qquad B_f^* = \frac{2 a h}{K_f} \tag{64}$$

(Note the factor 2 in the present definition of B_f^*)

From this point, the development is precisely the same as in VA1. Considering the counterparts of equations (35) and (36), precisely the same treatment is applicable. Figure 4, shows a graph similar to Figure 3. Figure 4 was actually constructed by computer analysis in order to show that the function ban (x , β) behaves like $\tan(x)$; the value of β used in constructing Figure 4 was $\beta = 0.9$. .

There is one difference which we would prefer to ignore. In the previous development, the fact that m had no non-real zeros or poles was proved by the argument contained in Appendix B. To construct a similar proof for the case in which tan x is replaced by ban (x, β) appears terribly formidable and we simply omit doing so.

Thus, a solution corresponding to (39) and expressed in exactly the same notation as (39) (that is, <u>identical to</u> (39)) is

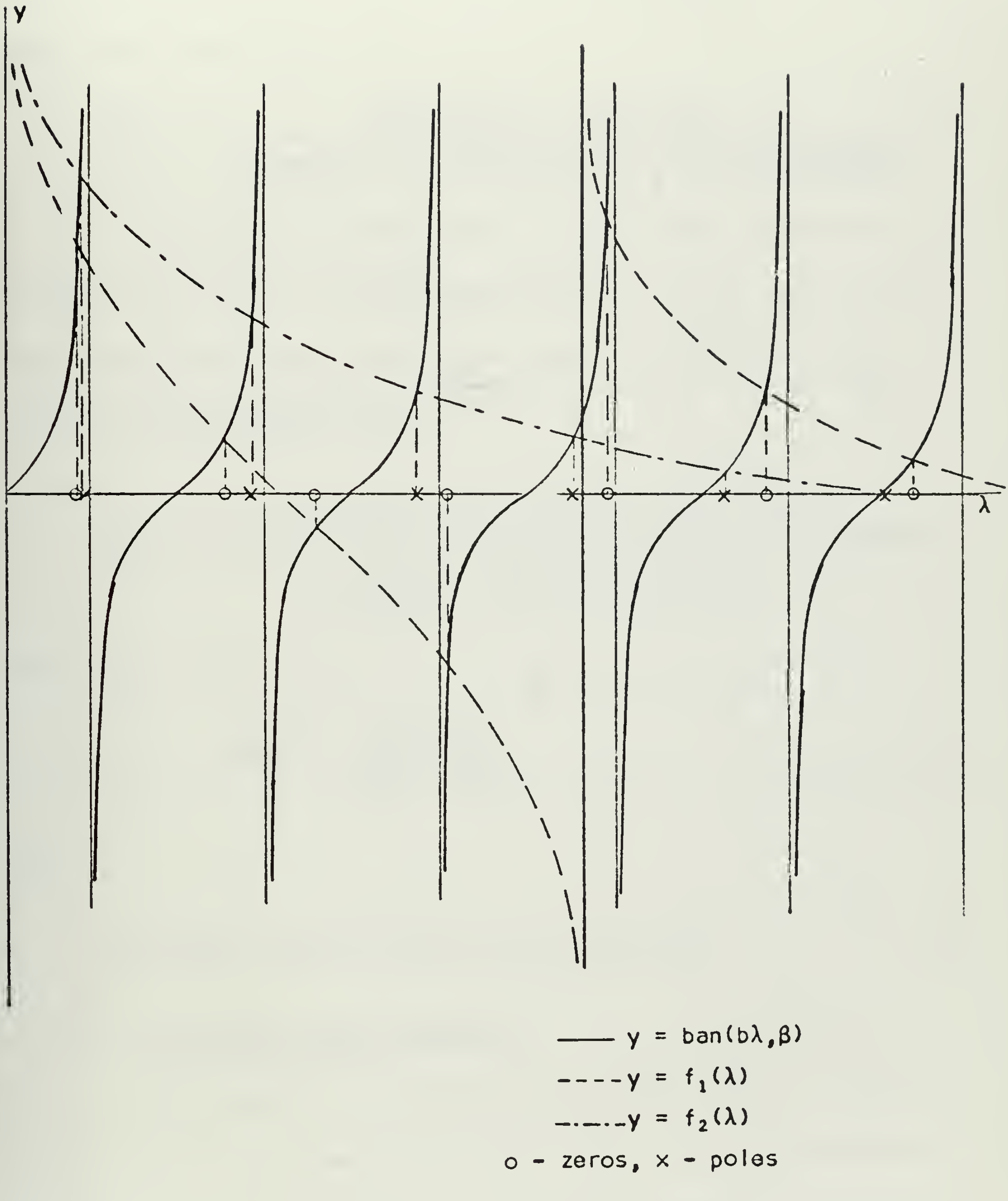


Figure 4: Poles and Zeros of m(s)
(Cylindrical Geometry)

applicable in the present case. However, the function $\bar{F}(r,s)$ in (28) must now be replaced by

$$\bar{F}(r,s) = \frac{I_0(\xi r)K_1(\xi b) + K_0(\xi r)I_1(\xi b)}{\left[-\frac{K_m}{h}\xi I_1(\xi a) + I_0(\xi a)\right]K_1(\xi b) + \left[\frac{K_m}{h}\xi K_1(\xi a) + K_0(\xi a)\right]I_1(\xi b)} \tag{65}$$

The inversion of (65) may be obtained using a solution discussed in Section 13.4 of [7], by first specializing to the case of a unit step input, and then by differentiating the result with respect to time. The result, corresponding to (40), is

$$T(r,x,t) = \pi\alpha_m K_m \sum_{n=1}^{\infty} \frac{\mu_n^2\left[{}_0\mathcal{C}_0(r,a;\mu_n) + \frac{K_m\mu_n}{h}\,{}_0\mathcal{C}_1(r,a;\mu_n)\right]e^{-\alpha_m\mu_n^2 t}}{\left[\frac{J_0(\mu_n a)}{J_1(\mu_n a)} + \frac{K_m\mu_n}{h}\right]^2 - \left[1 + \left(\frac{K_m\mu_n}{h}\right)^2\right]} \int_0^t e^{\alpha_m\mu_n^2\eta}\cdot\vartheta(x,\eta)\,d\eta \tag{66}$$

where μ_n are the roots of

$$\frac{K_m\mu^2}{h} = \frac{{}_0\mathcal{C}_1(a,b;\mu)}{{}_1\mathcal{C}_1(a,b;\mu)} \tag{67}$$

and

$${}_K\mathcal{C}_n(x,y;\beta) = J_K(\beta x)Y_n(\beta y) - Y_K(\beta x)J_n(\beta y) \tag{68}$$

2. <u>Asymptotic Approximations</u>

It is possible to proceed here in exactly the same way as in Section VA2. The principal difference is that in the present case of cylindrical geometry, in place of (43) we now define χ differently and arrive at different coefficients in the expansion. Thus

$$\chi = (bq)\,\mathrm{bank}(bq, a/b) = \frac{b}{2a}(b^2 - a^2)\frac{s}{\alpha_m} + \cdots \tag{69}$$

where ... denotes terms in s^2, etc. It requires painstaking work to establish that the coefficient of s^0 vanishes and that the first power term is as in (69) but the calculations are not illuminating and are not given here.

As in the case for slab geometry, the first non-trivial asymptotic expansion is obtained by retaining the first power of s in (69); correspondingly, we arrive at equation (41) the inverse of which is given in (48). In writing (47) we anticipated the present development by writing A_m/A_f in the final term of the radicand so as to be able to accommodate to both slab geometry and cylindrical geometry with one equation.

VI. A GENERALIZATION OF A PROBLEM CONSIDERED BY ARPACI

This case is described by (II-10) and (II-14) subject to conditions that will be stated below. The problem description is thus

$$\frac{\partial^2\theta}{\partial x^2} - \frac{V}{\alpha_f}\frac{\partial\theta}{\partial x} = \frac{1}{\alpha_f}\frac{\partial\theta}{\partial t} + \frac{hP}{A_f K_f}(\theta - T) \tag{1}$$

$$\frac{\partial T}{\partial t} = \frac{hP}{\rho_m c_m A_m}(\theta - T) \tag{2}$$

$$\theta = 1 \qquad x = 0 \quad , \quad t \geqq 0 \tag{3}$$

$$\theta = 0 \qquad x \to \infty \quad , \quad t \geqq 0 \tag{4}$$

$$\theta = 0 \qquad 0 \leqq x < \infty \quad , \quad t = 0 \tag{5}$$

$$T = 0 \qquad 0 \leqq x < \infty \quad , \quad t = 0 \tag{6}$$

$\theta = \theta(x, t)$, $T = T(x, t)$ are average temperatures in the sense of (II-7), (II-8), but we omit using the asterisk which distinguished this significance. The heat transfer coefficient h in (1) and (2) above is defined by (II-9).

This problem is a generalization of that considered in Section 7-5 of [2], the generalization coming about by the inclusion here of axial conductivity in the fluid, a mechanism not considered in Arpaci's analysis. This problem is a specialization of the problem considered

in Section IV hereof since here we neglect axial conduction in the duct wall, whereas this mechanism is included in the analysis given in Section IV.

Taking Laplace transformation with respect to time t, one obtains

$$\frac{\partial^2 \bar{\theta}}{\partial x^2} - \frac{V}{\alpha_f}\frac{\partial \bar{\theta}}{\partial x} - \left(\frac{s}{\alpha_f} + b_1\right)\bar{\theta} + b_1 \bar{T} = 0 \tag{7}$$

$$s\bar{T} = b_2(\bar{\theta} - \bar{T}) \tag{8}$$

where

$$b_1 = \frac{hP}{A_f K_f} \quad ; \quad b_2 = \frac{hP}{\rho_m c_m A_m} \tag{9}$$

Combining (7) and (8), one obtains

$$\frac{\partial^2 \bar{\theta}}{\partial x^2} - \frac{V}{\alpha_f}\frac{\partial \bar{\theta}}{\partial x} - \left(\frac{s}{\alpha_f} + b_1 - \frac{b_1 b_2}{s + b_2}\right)\bar{\theta} = 0 \tag{10}$$

subject to the conditions

$$\bar{\theta} = \frac{1}{s} \qquad x = 0 \tag{11}$$

$$\bar{\theta} \longrightarrow 0 \qquad x \longrightarrow \infty \tag{12}$$

Thus, the solution of (10) may be written

$$\bar{\theta} = \frac{1}{s}\exp\left[\frac{Vx}{2\alpha_f} - x\sqrt{\left(\frac{V}{2\alpha_f}\right)^2 + \frac{s}{\alpha_f} + b_1 - \frac{b_1 b_2}{s + b_2}}\right] \tag{13}$$

an increasing exponential having been rejected in order to satisfy (12). We write (13) as

$$\bar{\theta} = \frac{1}{s} \exp\left[\frac{Vx}{2\alpha_f} - \frac{x}{\sqrt{\alpha_f}}\sqrt{\frac{(s+\lambda)(s+\mu)}{(s+b_2)}}\right] \tag{14}$$

where

$$\lambda = \frac{\omega}{2}\left(1 + \sqrt{1 - \frac{V^2 b_2}{\alpha_f \omega^2}}\right) \tag{15}$$

$$\mu = \frac{\omega}{2}\left(1 - \sqrt{1 - \frac{V^2 b_2}{\alpha_f \omega^2}}\right) \tag{16}$$

$$\omega = b_2 + b_1 \alpha_f + \frac{V^2}{4\alpha_f} \tag{17}$$

It is easy to show that $\lambda > \mu > 0$. However, we want to indicate the magnitudes of λ and μ relative to b_2. To see this, consider the function

$$f(s) = \frac{s^2}{\alpha_f} + s\left(\frac{V^2}{4\alpha_f^2} + b_1 + \frac{b_2}{\alpha_f}\right) + \frac{b_2 V^2}{4\alpha_f^2} \tag{18}$$

the vanishing of which corresponds to the roots μ, λ ; viz. $f(-\lambda) = f(-\mu) = 0$. We note that the coefficient of the quadratic term is positive, and we note that

$$f(-b_2) = -b_1 b_2 < 0. \tag{19}$$

Thus, we easily see that,

$$\lambda > b_2 > \mu > 0 \tag{20}$$

Thus the condition of the theorem stated in Appendix A are satisfied, and we can write the inverse of (14) as

$$\theta(x,t) = 1 - \frac{1}{\pi}\exp\left(\frac{Vx}{2\alpha_f}\right)\left\{\int_{\mu}^{b_2} \frac{e^{-\eta t}}{\eta}\sin\left[x\,\rho(\eta)\right]d\eta + \int_{\lambda}^{\infty} \frac{e^{-\eta t}}{\eta}\sin\left[x\,\rho(\eta)\right]d\eta\right\} \tag{21}$$

where

$$\rho(\eta) = \sqrt{\frac{(\eta-\mu)(\lambda-\eta)}{\alpha_f(b_2-\eta)}} \tag{22}$$

Using the convolution theorem to invert (8), we obtain

$$T(x,t) = b_2\int_0^t e^{-b_2\eta}\,\theta(x, t-\eta)\,d\eta \tag{23}$$

We should remark that equations (1) and (2) above may be obtained by specializing equations (61) and (62) of Hiep's treatment [13]. However, we note again that Hiep's results depend upon numerical inversion of Laplace transformations, a process which our experience indicates is of doubtful accuracy and reliability.

VII. AN IMPROVEMENT ON A RESULT BY W. MUNK

Under the assumption that radial conduction is so efficient that essentially the same temperature prevails at all points of a particular fluid and metal cross section, W. Munk [24] obtained a very simple approximate solution for what he called "The Delayed Hot Water Problem." In the present section we obtain a very simple improvement over Munk's result. Hiep [13] has also obtained the improvement given here and he mentions that it has been obtained by others. Our reason for presenting it is that it is a natural deduction to be drawn from developments already made in this thesis.

We repeat equation (II-13) and the corresponding conditions.

$$\alpha_f \left(1 + \frac{K_m A_m}{K_f A_f}\right) \frac{\partial^2 T}{\partial x^2} - V \frac{\partial T}{\partial x} = \left(1 + \frac{K_m A_m \alpha_f}{K_f A_f \alpha_m}\right) \frac{\partial T}{\partial t} \tag{1}$$

$$T = 1 \quad \text{at} \quad x = 0 \quad , \quad t \geq 0 \tag{2}$$

$$T = 0 \quad \text{as} \quad x \to \infty \quad , \quad t \geq 0 \tag{3}$$

$$T = 0 \quad \text{at} \quad 0 \leq x < \infty \quad , \quad t = 0 \tag{4}$$

We have used T to denote the common value of average temperature. By defining

$$V^* = V \bigg/ \left(1 + \frac{\rho_m c_m A_m}{\rho_f c_f A_f}\right) \tag{5}$$

$$\alpha^* = \left(1 + \frac{K_m A_m}{K_f A_f}\right) \alpha_f \bigg/ \left(1 + \frac{\rho_m A_m c_m}{\rho_f A_f c_f}\right) \tag{6}$$

we may write

$$\alpha^* \frac{\partial^2 T}{\partial x^2} - V^* \frac{\partial T}{\partial x} = \frac{\partial T}{\partial t} \tag{7}$$

Taking Laplace transforms with respect to time, we obtain

$$\alpha^* \frac{\partial^2 \bar{T}}{\partial x^2} - V^* \frac{\partial \bar{T}}{\partial x} = s\bar{T} \tag{8}$$

The solution of (8) satisfying all conditions is

$$\bar{T} = \frac{1}{s} \exp\left[\frac{V^* x}{2\alpha^*} - x\sqrt{\left(\frac{V^*}{2\alpha^*}\right)^2 + \frac{s}{\alpha^*}} \right] \tag{9}$$

Inverting (9) by the use of standard tables [29] we obtain

$$T(x,t) = \frac{1}{2}\left[\operatorname{erfc}\left(\frac{x - V^* t}{2\sqrt{\alpha^* t}}\right) + e^{\frac{V^* x}{2\alpha^*}} \operatorname{erfc}\left(\frac{x + V^* t}{2\sqrt{\alpha^* t}}\right) \right] \tag{10}$$

This is the generalization described in the title of this section.

Munk's result is

$$T\ (x, t) = H\ (x - V^* t) \tag{11}$$

where H ( ) denotes the Heaviside unit step function. Thus the present result (10) is like Munk's in predicting a temperature wave front moving with a reduced velocity V^*, but, unlike Munk's it also provides for a softening of the wave front due to axial diffusion in fluid and pipe.

VIII. DISCUSSION

A. SCOPE OF ANALYSIS

The general problem as formulated by equations (II-1) and (II-2) for the case of cylindrical geometry and equations (II-3) and (II-4) for the case of slab geometry, even with the simplifying assumptions of subsection IIA2, are mathematically very involved. These equations contain two dependent variables, namely the fluid and the pipe wall temperature each of which is a function of three independent variables. There is also a formidable number of boundary and initial conditions to be satisfied. In Section III the problem is reduced to the solution of an integral equation the only unknown in which is the temperature at the interface, which is a function of only two variables; no auxiliary conditions are required. However, as may clearly be observed by the reader, the integral equation of interest is mathematically very involved, and retains a great share of the complexity of the original problem, for the following reasons.

Firstly, the unknown function appears only under the integral sign (the so called integral equation of the first type). Secondly, there are two integral signs under which the function of interest appears. Thirdly, one of the integrals has a semi-infinite range, namely o to ∞. Fourthly, the kernels of integrations are complicated functions.

The complexity of the general problem, either in its original form or as compressed into this integral equation is such as to have led us to focus attention upon further simplifications which are more tractable from a mathematical viewpoint and the analyses of which are treated in other sections of this thesis. However, the integral equation formulation is believed to be of significance and its treatment, probably by numerical procedure, is considered feasible.

Further simplifying assumptions are introduced in IID, in order to reduce the mathematical difficulties. In a search for the various methods which would render the general problem amenable to mathematical analysis, it is found that by defining a suitable mean temperature in the direction perpendicular to the direction of flow, and by making a further assumption of constant surface heat transfer coefficient, the useful formulation given in (II-10) and (II-11) is obtained.

Assumption of large velocity justifies the use of a perturbation technique, which is presented in Section IV hereof.

The zeroth order terms, which corresponds to a solution given in Section 15-3 of [7] and also in example 7.5 of [2],are obtained, and first corrections to the fluid and pipe temperatures are obtained explicitly. These solutions are found not to be valid at and beyond the temperature front. Accordingly a singular perturbation technique is used to obtain an expression which traces the behavior ahead of the front.

In Section VI a further physical assumption, namely neglecting axial conduction in the solid, leads to a problem for which we are able to find the closed form solution by using the method of Laplace transformation. This affords an improvement over the solutions given in [7] and [2], cited in the preceding paragraph.

In Section V the physical assumptions that are introduced consist of considering an average fluid temperature and neglecting axial conduction in the solid. This case is solved using the method of Laplace transformation. Even though the methods of transform calculus reduce the partial differential equations to a system which can be readily solved in the transform domain, sometimes delicate mathematical handling is required to invert the solutions back to the real variable domain. Such a case is that of equation (V-27) and (V-28), where the solutions in transformed form contain an infinite number of singularities (branh points). An elegant theorem for this inversion, established by Professor J. E. Brock, is presented in Appendix A. Thus closed form solutions are obtained. Furthermore, use is made of the properties of Laplace transform to yield some useful asymptotic solutions.

In Section VII the view taken is that of W. Munk [24]. It was found possible to improve upon his solution by accounting for the axial conduction in the fluid as well as the solid wall. This treatment assumes, as Munk did, an instantaneous common constant uniform temperature at any cross section of the fluid and wall. Although

this result is obtained independently, it turns out to be known; see Appendix II of [13]. This solution is useful for small fluid velocities; Cf.Hiep [13], p. 32 and Preston [27].

We will make no further attempt to relate the problems considered herein to practical engineering situations. Such references as [7] and [2] are devoted to problems having similar relations to practical engineering situations. Their importance lies in the fact that they do provide definite mathematical solutions to well posed mathematical problems which have relation to the physical world of heat transfer engineering; it is the obligation of the person who employs such results to investigate the degree to which the mathematical conditions are satisfied in his own physical application.

B. RECOMMENDATIONS

Most important is treatment of the integral equation developed in Section III. Remarks about this are included in subsection IIIF.

It would be useful to develop asymptotic formulas to employ in place of the exact inversion in cases where there are severe numerical difficulties associated with the latter. The method of steepest descents (e.g. Cf. Carrier Crook and Pearson - Section 6-3 [8]), appears promising for this purpose.

There are several mathematical details which should be filled in (although in treatments similar to our own they are usually glossed over). These all relate to locations of the zeros and poles of

transcendental functions. In the case of equations (V-59) and (III-155), we have simply assumed that the zeros and poles are all real; under this assumption they are not particularly difficult to locate. However a proof that there are no complex zeros or poles is really required.

Inasmuch as instantaneously higher rates of heat transfer are to be expected at and near a rapidly moving front, it might be expected that an analysis employing finite speed of heat propagation would be enlightening. Accordingly, it is suggested that such an analysis be attempted in one of the simpler subcases dealt with herein; Cf., for examples, Section 1-6 of [10].

Also since numerical difficulties are to be expected in applying the inversion theorem of Appendix A to the heat transfer problems herein, it is suggested that actual numerical calculations be undertaken. A multiple-precision facility, such as described in [14] and [28] seems necessary for this purpose.

Another thing which comes to mind is to relax the specialization herein because of the assumption of slug flow, i. e., constant fluid velocity. It seems clear that the assumption of velocity as a given function of the radial or transverse dimension would provide definite differential systems for analysis, but that these would, in general, be much more difficult to deal with than those considered herein.

The treatment given by Hiep [13] relies heavily on the accuracy and validity of a numerical inversion of Laplace transform due to

Salzer [30], [31], [32]. However our experimentation with such methods of inversion showed that reliably accurate results could be obtained only in limited ranges and for relatively simple functions. This persuaded us to look for other analytical methods of treating the problem, such as perturbation method treated in Section IV hereof. However, it is probable that further development of techniques of numerical inversion may provide a useful and powerful tool for analyses such as those studied herein.

We should also mention our feeling that no great difficulty would be associated with inferring response to a unit step change in left end temperature but retaining the other conditions specified in the analysis of Sparrow and DeFarias [33], who employed a sinusoidally varying left end boundary condition.

Also although throughout we have assumed that the conduit had perfect exterior insulation so that there was no heat flux through this surface, it seems clear that other exterior boundary conditions (such as an isothermal condition) would, in most applications considered herein, involve only changes in detail and not essential changes in procedure.

APPENDIX A

INVERSION OF SOME LAPLACE TRANSFORMS

The theorem stated and proved in this appendix provides the Laplace inversion of (V-27), (V-28), and (VI-14) of the text. This appendix is largely a paraphrase of [6].

<u>THEOREM:</u> If the function to be inverted is of the form

$$f(s) = \frac{1}{s} \exp\left[-\sqrt{m(s)}\right] \tag{1}$$

where m(s) is a meromorphic function whose zeros $s = -a_k$ and whose poles $s = -b_k$ satisfy

$$0 < a_1 < b_1 < a_2 < b_2 \cdots < a_N < b_N \cdots < \infty \tag{2}$$

Then, for $t > 0$, the inverse Laplace transform $F(t) = \mathcal{L}^{-1}\left[f(s)\right]$ is given by

$$F(t) = \exp\left[-\mu(0)\right] - \frac{1}{\pi}\sum_{k=1}^{N}\int_{a_k}^{b_k} \frac{e^{-\eta t}}{\eta} \sin\left[\mu(\eta)\right] d\eta \tag{3}$$

where

$$\mu(\eta) = \sqrt{|m(-\eta)|} \tag{4}$$

The interpretation of the sum in (3) is as follows. If there is a finite number N of pairs $a_k < b_k$, no special interpretation is necessary. If there is a finite number (N-1) of pairs $a_k < b_k$ and an

additional zero $s = -a_N$, $a_N > b_{N-1}$, an N^{th} pole $s = -b_N = -\infty$ is to be added so as to make N pairs. If there is an infinite number of pairs $a_k < b_k$, take $N = \infty$.

In proving the theorem stated above we consider a sequence $\{R_n\}$ of positive numbers, $R_n \neq a_k$, b_k for any n and any k, $R_n \to \infty$ as $n \to \infty$, and we consider a corresponding sequence of closed contours C_n in the complex s- plane each of which encloses the pole of f(s) at $s = 0$ but excludes all of the branch points at $s = -a_k$ and $s = -b_k$. The contour C_n consists of four subcontours C_{n1}, C_{n2}, C_{n3} and C_{n4} as follows (see Figure A-1). C_{n1} is a vertical line from $s = -iR_n$, to $s = iR_n$, indented as shown by a small semicircle passing to the right of $s = 0$. C_{n2} is the arc $s = R_n e^{i\theta}$, $\frac{\pi}{2} \leq \theta \leq \pi$ and C_{n4} is the similar arc $s = R_n e^{i\theta}$, $-\pi \leq \theta \leq -\frac{\pi}{2}$. Finally C_{n3} is the contour which passes to the right from $s = -R_n$, along the negative real s-axis to $s = -a_1$ with small semicircular indentations into the upper half s - plane about each zero $(-a_k)$ and pole $(-b_k)$ and then passes to the left from $s = -a_1$ to $s = -R_n$ with small semicircular indentations into the lower half s - plane, as indicated in Figure A-1.

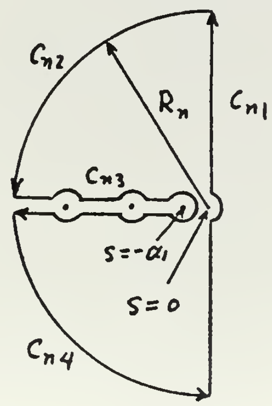


Figure A-1: Contour C_n

Clearly by Cauchy's theorem

$$I_n = \frac{1}{2\pi i}\oint_{C_n} e^{st} f(s)\,ds = \exp\left[-\sqrt{m(0)}\right] = \exp\left[-\mu(0)\right] \tag{5}$$

for n = 1, 2, 3 . . . . Also, by the complex inversion theorem for Laplace transforms

$$I_{n1} = \frac{1}{2\pi i}\int_{C_{n1}} e^{st} f(s)\,ds \longrightarrow F(t) \tag{6}$$

as $n \longrightarrow \infty$.

To proceed further we wish to establish an inequality for arg [m(s)]. To do this we consider complex vectors emanating from the points $s = -a_k$ and $s = -b_k$, and we write

$$1 + \frac{s}{a_k} = u_k \exp(i\phi_k) \; ; \qquad u_k \geqq 0 \quad ; -\pi \leqq \phi_k \leqq \pi \tag{7}$$

$$1 + \frac{s}{b_k} = v_k \exp(i\psi_k) \; ; \qquad v_k \geqq 0 \quad ; -\pi \leqq \psi_k \leqq \pi \tag{8}$$

$$m_n(s) = m(0) \prod^{(n)} \left(1 + \frac{s}{a_k}\right) \div \prod^{(n)} \left(1 + \frac{s}{b_k}\right) \tag{9}$$

$$= \mu_n^2(s) \exp(i\beta_n) \tag{10}$$

where

$$\mu_n^2(s) = \mu(0) \prod^{(n)} u_k \div \prod^{(n)} v_k \tag{11}$$

and

$$\beta_n = \sum^{(n)} \phi_k - \sum^{(n)} \psi_k \tag{12}$$

The sums and products written with a superscript (n) indicate that only those terms are to be considered for which $a_n < R_n$ or $b_n < R_n$, as may be appropriate.

Now note for any s in the (closed) upper half plane we have

$$\phi_K \geqq \psi_K \quad , \qquad K = 1, 2, \dots \tag{13}$$

A geometrical argument is given in Figure A-2. Now either there are N pairs a_k, b_k each less than R_n, so that

$$\beta_n = \sum_{K=1}^{N} (\phi_K - \psi_K) \quad \geqq 0 \tag{14}$$

or there is an extra, unpaired zero, so that

$$\beta_n = \sum_{K=1}^{N-1} (\phi_K - \psi_K) + \phi_N \geqq 0 \tag{15}$$

In the first case we can write

$$\begin{aligned} \beta_n &= \phi_1 - \sum_{K=1}^{N-1} (\psi_K - \phi_{K+1}) - \psi_N \\ &\leqq \phi_1 - \psi_N < \phi_1 \leqq \pi \end{aligned} \tag{16}$$

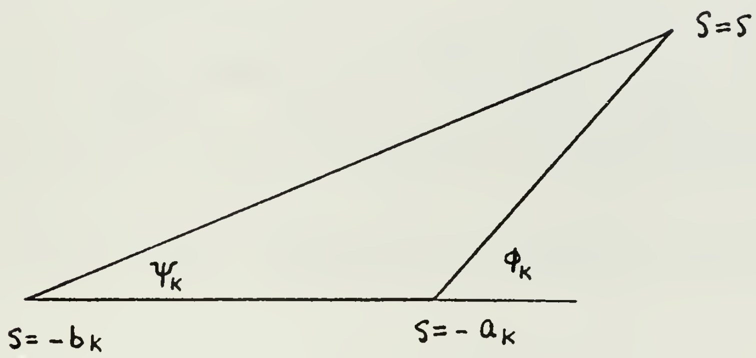


Figure A-2: Demonstration that $\phi_K \geqq \psi_K$

whereas in the second case we can write

$$\beta_n = \phi_1 - \sum_{K=1}^{N} (\psi_K - \phi_{K+1}) \leqq \phi_1 \leqq \pi \tag{17}$$

Thus in either case

$$0 \leqq \beta_n \leqq \pi \tag{18}$$

Similarly by considering an s in the lower half plane we find

$$-\pi \leqq \beta_n \leqq 0 \tag{19}$$

so that in all cases

$$-\pi \leqq \beta_n \leqq \pi \tag{20}$$

Thus, if we write

$$I_{n2} = \frac{1}{2\pi i} \int_{C_{n2}} e^{st} f(s)\, ds \tag{21}$$

we note that

$$\begin{aligned} |f(s)| &= \frac{1}{R_n} \left| \exp\left(-\sqrt{\mu_n^2 e^{i\beta_n}}\right) \right| \\ &= \frac{1}{R_n} \exp\left[-\mu_n \cos(\beta_n/2)\right] \leqq \frac{1}{R_n} \end{aligned} \tag{22}$$

uniformly on C_{n2}, by virtue of the inequality (20). Thus, by Jordan's Lemma $I_{n2} \longrightarrow 0$ as $n \longrightarrow \infty$. Similarly the integral on C_{n4} vanishes as $n \longrightarrow \infty$.

Next we consider the integrals around semicircles surrounding poles and zeros on contours C_{n3}. The result is obvious near a zero $s = -a_k$; $e^{st} \approx e^{-a_k t}$, $|f(s)| \approx \frac{1}{a_K}$ since $m(s) \approx 0$, and

the length of the arc goes to zero as its radius does. Thus the integral on such semicircles about a zero, $s = -a_k$ vanishes with the radius. However, for the integral on a semicircle about a pole, $s = -b_k$, we must invoke (20) again. We have

$$\left| \exp\left[-\sqrt{m(s)}\right] \right| = \exp\left[-\mu_n \cos\left(\beta_n/2\right)\right] < 1 \tag{23}$$

even though $\mu_n \longrightarrow \infty$ as the radius of the indentation goes to zero. Thus the integral on such a semicircle about a pole, $s = -b_k$, vanishes with the radius.

We remark that although we have reason to believe, on physical grounds, the correctness of an evaluation by Kantola [18], we feel that his argument intended to show vanishing of such an integral on a semicircle surrounding a branch point is deficient. Our argument, based on (20), is not applicable to his case.

If now we write (5) in the form

$$I_{n1} = I_n - I_{n2} - I_{n3} - I_{n4} \tag{24}$$

and let $n \longrightarrow \infty$, and also let the radii of all semicircular indentations on C_{n3} go to zero, we find

$$F(t) = \exp\left[-\mu(0)\right] - I_3 \tag{25}$$

where

$$I_3 = \frac{1}{2\pi i} \int_{C_3} \frac{1}{s} \exp\left(st - \sqrt{m(s)}\right) ds \tag{26}$$

and C_3 now is a contour extending from $s = -\infty$ along the upper edge of a branch cut, to the right, passing around $s = -a_1$, and then extending along the lower edge of the branch cut to $s = -\infty$. The branch cut is the negative real s - axis from $-\infty$ to $-a_1$. There are no longer any semicircular identations in C_3.

To reduce (26) to a more convenient form, we first introduce the function $\nu(x)$, for real x, which denotes the excess of the number of zeros ($s = -a_k$) over the number of poles ($s = -b_k$) to the right of the point $s = x$. It is clear that $\nu = 0$ in the interval $-a_{k+1} < s < -b_k$ while $\nu = 1$ in the interval $-b_k < s < -a_k$. Writing β now, rather than β_n, since we have taken $n \longrightarrow \infty$ and all zeros and poles are included in our accounting, we note that on the upper edge of the cut $\beta = \nu\pi/2$ while on the lower edge $\beta = -\nu\pi/2$. We also now write μ rather than μ_n. Thus

$$
\begin{aligned}
I_3 = \frac{1}{2\pi i}\Bigg\{ & \int_{-\infty}^{-a_1} \frac{1}{s} \exp\left[st - \mu \exp(i\nu\pi/2)\right] ds \; + \\
& \int_{-a_1}^{-\infty} \frac{1}{s} \exp\left[st - \mu \exp(-i\nu\pi/2)\right] ds \Bigg\} \qquad (27) \\
= \frac{1}{2\pi i} & \int_{-\infty}^{-a_1} \frac{1}{s} e^{st} \Big\{ \exp\left[-\mu \exp(i\nu\pi/2)\right] - \exp\left[-\mu \exp(-i\nu\pi/2)\right] \Big\} ds
\end{aligned}
$$

If $\nu = 0$, the quantity in braces vanishes, while if $\nu = 1$ it becomes

$$\exp(-i\mu) - \exp(i\mu) = -2i \sin\mu \qquad (28)$$

Thus

$$I_3 = -\frac{1}{\pi}\sum_{K=1}^{N}\int_{-b_K}^{-a_K}\frac{e^{st}}{s}\sin\left[\mu(s)\right]ds \tag{29}$$

If we now change the variable of integration from s to $\eta = -s$, we obtain

$$I_3 = \frac{1}{\pi}\sum_{K=1}^{N}\int_{a_K}^{b_K}\frac{e^{-\eta t}}{\eta}\sin\left[\mu(\eta)\right]d\eta \tag{30}$$

and, thus, finally

$$F(t) = \exp\left[-\mu(0)\right] - \frac{1}{\pi}\sum_{K=1}^{N}\int_{a_K}^{b_K}\frac{e^{-\eta t}}{\eta}\sin\left[\mu(\eta)\right]d\eta \tag{31}$$

The use of the upper limit of summation is consistent with the remarks following the statement of the theorem, the proof of which is now complete.

In equation (VI-14) the solution obtained in Laplace transform form is a special case in which we have

$$m^2(s) = k^2 (s+a_1)(s+a_2)\Big/(s+b_1) \tag{32}$$

where $k^2 > 0$ and $0 < a_1 < b_1 < a_2$. The numbers a_1, a_2 and b_1 are known explicitly by (VI-9, 15, 16). We have

$$\mu(0) = \mu_0 = k\sqrt{a_1 a_2 / b_1} \tag{33}$$

and the inversion is

$$F(t) = e^{-\mu_0} - \frac{1}{\pi}\int_{a_1}^{b_1} \frac{e^{-\eta t}}{\eta} \sin\left[\mu(\eta)\right] d\eta - \frac{1}{\pi}\int_{a_2}^{\infty} \frac{e^{-\eta t}}{\eta} \sin\left[\mu(\eta)\right] d\eta \qquad (34)$$

APPENDIX B

NONEXISTENCE OF COMPLEX ROOTS OF EQUATION (V-36)

Demonstration that such equations as (V-36) and (V-38) have only real roots are occasionally called for in order to employ Cauchy's Theorem (and Jordan's Lemma) to evaluate the inverse Laplace transforms encountered in heat conduction problems. Carslaw and Jaeger [7] have some general remarks on this matter (Section 12.8 IV , page 324), making use of Sturmian theory; their reference is to Chapter X of [15]. Polya and Szego [26] consider similar problems, most of them concerned with the zeros of polynomials, but some,e. g., Auf.172, of nature similar to ours. They use various methods of proof. Professor J. E. Brock has constructed the following elementary proof for the case encountered in equation (V-36). The case encountered in equation (V-61) wherein the simple trignometric tangent function is replaced by a generalization involving quotients of expressions involving J- and Y- Bessel functions, is obviously much more difficult and we simply make no attempt to construct a proof in this case, taking refuge in the (weak) argument that the function we encounter is "similar" to the tangent function.

Thus, we consider

$$x\tan x = \frac{c(n-\delta-x^2)}{n-x^2} \qquad c>0\,, \quad n>0 \quad 0\leqq\delta<n \qquad (1)$$

Let $x = a + ib$, a, b real and presume $b \neq 0$

We will show that this leads to a contradiction

$$\tan(a + ib) = (\tan a + i \tanh b)/(1 - i \tan a \tanh b).$$

Denoting the left side of (1) by L, then

$$L = x \tan x = (a + ib)\left(\frac{\tan a + i \tanh b}{1 - i \tan a \tanh b}\right)\left(\frac{1 + i \tan a \cdot \tanh b}{1 + i \tan a \cdot \tanh b}\right) \qquad (2)$$

which may be written as

$$x \tan x = \frac{(a \sin 2a - b \sinh 2b) + i(b \sin 2a + a \sinh 2b)}{D_1} \qquad (3)$$

where

$$D_1 = \cos 2a + \cosh 2b.$$

Next considering the right hand side of equation (1) and denoting it

by R we obtain

$$R = \frac{c\left[(n - \delta - a^2 + b^2) - 2iab\right]}{(n - a^2 + b^2) - 2iab} \cdot \frac{(n - a^2 + b^2) + 2iab}{(n - a^2 + b^2) + 2iab}$$
$$= \frac{(n + b^2 - a^2)^2 + 4a^2b^2 - \delta(n^2 + b^2 - a^2) - 2iab\delta}{D_2} \qquad (4)$$

where

$$D_2 = \left[(n + b^2 - a^2) + 4a^2b^2\right]/c \qquad (5)$$

At this point it is well to observe that for $b \neq 0$, $\cosh b > 1$

While for any real a, $\cos a \geq -1$

and thus we conclude that

$$D_1 > 0 \qquad (6)$$

Also

$$D_2 = \left| \frac{(n-a^2+b^2)-2iab}{c} \right| > 0 \qquad n>0 \quad , \quad b^2>0 \tag{7}$$

for $(n - a^2+b^2)$ and $(2ab)$ cannot both vanish.

Now if we equate the real and imaginary parts of (3) and (4), we obtain

$$\frac{a \sin 2a - b \sinh 2b}{D_1} = \frac{(n+b^2-a^2)^2+4a^2b^2-\delta(n+b^2-a^2)}{D_2} \tag{8}$$

by equating the real parts, and

$$\frac{\dfrac{\sin 2a}{a} + \dfrac{\sinh 2b}{b}}{D_1} = -\frac{2\delta}{D_2} \tag{9}$$

by equating the imaginary parts. From (9) above we may write

$$\frac{a \sin 2a}{D_1} = -a^2 \frac{\sinh 2b}{b D_1} - \frac{2\delta a^2}{D_2} \tag{10}$$

and substituting in (8) we obtain

$$-(a^2+b^2)\frac{\sinh 2b}{b D_1} = \frac{(n+b^2-a^2)^2+4a^2b^2-\delta(n+b^2-a^2)+2\delta a^2}{D_2} \tag{11}$$

Now considering the numerator on the right of (11). Call it N, and consider it as a function of δ ; then

$$\frac{dN}{d\delta} = -(n+b^2-a^2) + 2a^2 = 3a^2-n-b^2 = \rho \tag{12}$$

First, supposing that $\rho > 0$, then the minimum value of N occurs for $\delta = 0$.

$$N_{min} = (n + b^2 - a^2)^2 + 4a^2b^2 > 0$$

Next suppose that $\rho < 0$, then the minimum value of N occurs for $\delta = n$.

$$\begin{aligned} N_{min} &= (n + b^2 - a^2)^2 + 4a^2b^2 - n(n + b^2 - a^2) + 2n\,a^2 \\ &= n(a^2 + b^2) + (a^2 + b^2)^2 > 0 \end{aligned} \tag{13}$$

Thus in any event $N > 0$, and since for any real $b \neq 0$ we have

$$\frac{\sinh 2b}{b} > 0 \ , \tag{14}$$

thus the left side of (11) is less than zero whereas the right side is greater than zero; which is a contradiction.

Thus we cannot have roots of the form $x = a + ib$ with a and b real, and $b \neq 0$. Therefore we can only have real roots.

BIBLIOGRAPHY

1. Abramowitz, M., and Stegun, I. A., Handbook of Mathematical Functions, U. S. Dept. of Commerce, National Bureau of Standards, Applied Mathematics Series, No. 55, 1964.

2. Arpaci, V. S., Conduction Heat Transfer, Addison-Wesley Publishing Company, 1966.

3. Arpaci, V. S., "Transient Conduction in Coaxial Cylinders with Relative Motion and Heat Conduction," Journal of Applied Mechanics, v. 27, p. 623-628, 1960.

4. Avdonin, N. A., "A Two-Zone Problem in Convective Heat Transfer," Soviet Physics-Doklady, v. 8, no. 8, p. 782-784, 1964.

5. Bickley, W. G., Bessel Functions and Formulae, University Press Cambridge, 1966.

6. Brock, J. E., Zargary, A. and Comstock, C., Inversion of Some Laplace Transforms Encountered in Heat Transfer Problems. To be published as a U. S. Naval Postgraduate School Technical Note.

7. Carslaw, H. S., and Jaeger, J. C., Conduction of Heat in Solid, 2d ed. Oxford at the Clarendon Press, 1959.

8. Carrier, G. F., Krook, M., and Pearson, C. E., Functions of a Complex Variable, McGraw-Hill, 1966.

9. Cole, J. D., Perturbation Methods in Applied Mathematics, Blaisdel Publishing Company, 1968.

10. Eckert, E. R. G., and Drake, R. M., Jr., Analysis of Heat and Mass Transfer, McGraw-Hill, 1972.

11. Graetz, L., Uber die Warmeleitfahigheit von Flussigkeiten," Ann. Phys., 18, 79 (1883), 25, 337 (1885).

12. Groebner, W. and Hofreiter, N., Integraltafel, Springer-Verlag, v. I, 1965; v. II, 1966.

13. Hiep, D. D., Heat Transfer in Porous Media, M. S. Thesis, U. S. Naval Postgraduate School, Monterey, California, 1965.

14. IBM, Multiple-Precision Floating-Point Arithmatic Package, Document No. 360D-40.4.003, Share Programme Library Agency, IBM Corporation, White Plains, New York.

15. Ince, E. L., Ordinary Differential Equations, Dover Publications, New York, 1944.

16. Jakob, M., Heat Transfer, v. 1, Wiley, 1949.

17. James, D. E., and William, N. G., "The Effect of Axial Conduction in the Wall on Heat Transfer with Laminar Flow," Int. J. Heat Mass Transfer, v. 13, p. 452-470, 1970.

18. Kantola, R. A., Transient Response of Fluid Lines, Including Frequency Modulated Inputs, Ph-D Thesis, Polytechnic Institute of Brooklyn, 1969.

19. Kelleher, M. D., and Yang, K. T., "A Steady Conjugate Heat Transfer Problem with Conduction and Free Convection". Appl. Sci. Research, v. 17, p. 249-269, 1967.

20. Lowan, A. N., "On Some Two-Dimensional Problems in Heat Conduction," Phil. Mag. and J. of Applied Science, v. 24, 7th ser, p. 410-424, July-Dec. 1937.

21. Lowan, A. N., "Notes on the Problem of Heat Conduction in a Semi-Infinite Hollow Cylinder," Quarterly of Applied Mathematics, v. 2, p. 348-350, 1945.

22. Luikov, A. V., Aleksashenko, V. A., and Aleksachenko, A. A., "Analytical Methods of Solution of Conjugated Problems in Convective Heat Transfer," Int. J. Heat Mass Transfer, v. 14, p. 1047-1056, 1971.

23. Moreland, F. E., Solution of the Single Blow Problem with Longitudinal Conductivity by Numerical Inversion of Laplace Transform, M. S. Thesis, U. S. Naval Postgraduate School, Monterey, California, 1964.

24. Munk, W., "The Delayed Hot Water Problem," J. Applied Mech., p. 193, June 1954.

25. Perelman, T. L., "On Conjugated Problems of Heat Transfer," Int. J. Heat Mass Transfer, v. 3, p. 293-303, 1961.

26. Polya, G. and Szego, G. Aufgaben und Lehrsatze aus der Analysis, Vol. II, Dover Publications, New York, 1945.

27. Preston, F. W., Ph-D Thesis, Pennsylvania State University, 1957.

28. Rainey, S. D., Multiple Precision Arithmetic Package, U. S. Naval Postgraduate School Technical Note. No. 0211-19, May 1972.

29. Roberts, G. E. and Kaufman, H., Table of Laplace Transforms, W. B. Saunders Company, Philadelphia, 1966.

30. Salzer, H. E., "Orthogonal Polynomials Arising in Numerical Evaluation of Inverse Laplace Transforms," Math. Tables Aids Computer. 9, p. 164-177, 1955.

31. Salzer, H. E., "Additional Formulas and Tables for Orthogonal Polynomials Originating from Inversion Integrals," J. Math. Phys. v. 40, p. 72-86, 1961.

32. Salzer, H. E., "Table for the Numerical Calculation of Inverse Laplace Transforms," J. Math and Phys., p. 89-108, 1958.

33. Sparrow, E. M. and F. N. De Farias, "Unsteady Heat Transfer in Ducts with Time Varying Inlet Temperature and Participating Walls," Int. J. Heat Mass Transfer, v. 11, p. 837-853, 1968.

34. Siegel, R. "Forced Convection in a Channel with Wall Heat Capacity and with Wall Heating Variable with Axial Position and Time," Int. J. Heat Mass Transfer, v. 6, p. 607-620, 1963.

35. Siegel, R. and J. M. Savino, "An Analytical Solution of the Effect of Peripheral Wall Conduction on Laminar Forced Convection in Rectangular Channels," Trans, ASME Paper No. 64-HT-24, September 1964.

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13. ABSTRACT

Transient temperature distribution in axisymmetric systems of incompressible fluid flow through semi-infinite, externally insulated cylindrical pipes is considered under the assumption of constant uniform velocity and constant material properties. The general problem, including all significant heat transfer mechanisms, except radiation, is reduced to an integral equation containing only one unknown function, namely an interface temperature as a function of time and axial position. By neglecting one or more heat transfer mechanisms, several other formulations are obtained. These are analyzed so as to provide approximate and closed form solutions not previously available. Similar formulations and results are obtained for the case of fluid between parallel plates.

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Heat transfer						
Conjugated heat transfer						
Transient heat transfer						
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Perturbation						
Integral equation						